Concave and convex functions

111 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-concave-convex-points-of-inflection/

Date

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A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Given $f(x) = x^4 - 4x^3$, find $f''(x)$.

Answer .....

A2 Needed today [2 marks]

Given $f(x) = xe^{2x}$, find $f''(x)$. Give it in the form $e^{2x}(a + bx)$.

Answer .....

A3 Needed today [2 marks]

Solve the inequality $x^2 - 4x + 3 < 0$.

Answer .....

A4 Common slip [2 marks]

For $y = x^4$, $\dfrac{d^2y}{dx^2} = 0$ when $x = 0$. Is the origin a point of inflection? Give a reason.

Answer .....

A5 From chapter 9 [2 marks]

Find the coordinates of the stationary point on $y = x^2 - 6x + 1$ and say whether it is a maximum or a minimum.

Answer .....

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B Example, then your turn

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Find the interval on which $f(x) = x^3 - 6x^2 + 5x$ is concave, and the interval on which it is convex.

Hint Concave where $f''(x) \leqslant 0$, convex where $f''(x) \geqslant 0$.

B1 Find the set of values of x for which $f(x) = x^4 - 6x^2 + 3x$ is concave, and the set for which it is convex. [4 marks]

Answer .....

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EXAMPLE 2

Show that the curve $y = x^3 - 3x^2 + 4$ has one point of inflection, and find its coordinates.

Hint Solve $\frac{d^2y}{dx^2} = 0$, then check that the sign changes on either side.

B2 Find the coordinates of the point of inflection on the curve $y = 2x^3 + 3x^2 - 12x + 1$. Show that the second derivative changes sign there. [4 marks]

Answer .....

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EXAMPLE 3

Show that $f(x) = x \ln x$ is convex for all $x > 0$.

Hint Convex for all x means $f''(x) \geqslant 0$ for every x in the domain.

B3 Show that $f(x) = \ln x - x^2$ is concave for all $x > 0$. [4 marks]

EXAMPLE 4

Find the coordinates of the point of inflection on $y = xe^{-x}$.

Hint Use the product rule twice and keep e^{-x} as a factor. e^{-x} is never zero.

B4 Find the exact coordinates of the point of inflection on $y = (x + 1)e^{-x}$. [5 marks]

Answer .....

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EXAMPLE 5

Find and classify the stationary points of $y = x^4 - 4x^3$, and find any other point of inflection.

Hint At a stationary point where $\frac{d^2y}{dx^2} = 0$, look at the sign of $\frac{dy}{dx}$ either side.

B5 Find and classify the stationary points of $y = x^5 - 5x^4$, and find the coordinates of the point of inflection. [6 marks]

Answer .....

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C1 Asha says: "For $y = x^4 - 4x$, $\dfrac{d^2y}{dx^2} = 0$ at $x = 0$, so $(0, 0)$ is a point of inflection." Explain why Asha is wrong, and say whether the curve is concave or convex. [4 marks]

C2 $f(x) = x^3 + 3x^2 - 9x$. Find the set of values of x for which f is convex, and the coordinates of the point of inflection. [4 marks]

Answer .....

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C3 Find the coordinates of the points of inflection on $y = \sin x + \cos x$ for $0 \leqslant x \leqslant 2\pi$. [5 marks]

Answer .....

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C4 Ben differentiates $y = x^2 \ln x$ and writes $\frac{dy}{dx} = 2x \times \frac{1}{x} = 2$, so $\frac{d^2y}{dx^2} = 0$ and "the curve has no points of inflection." Explain his mistake, and find the exact coordinates of the point of inflection. [5 marks]

Answer .....

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D1 A curve has equation $y = x^3 - 6x^2 + 9x + 2$.

(a) Find the coordinates of the stationary points, and determine their nature. [4 marks]

Answer .....

(b) Find the coordinates of the point of inflection. [2 marks]

Answer .....

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(c) Find the set of values of x for which the curve is both concave and increasing. [3 marks]

Answer .....

Total for D1: 9 marks

D2

> Sam is studying $f(x) = 3x^5 - 10x^4$. He writes: "$f''(x) = 0$ at $x = 0$ and at $x = 2$, so the curve has two points of inflection."

Sam is wrong.

(a) Show that $f''(x) = 60x^2(x - 2)$. [2 marks]

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(b) Explain why the origin is not a point of inflection. [3 marks]

(c) Find the coordinates of the point of inflection, and the set of values of x for which f is convex. [4 marks]

Answer .....

Total for D2: 9 marks

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D3 The curve $y = x^4 + ax^3 + 6x^2$ has no points of inflection. Find the set of possible values of the constant a. [7 marks]

Answer .....

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D4 A curve has equation $y = e^{-x^2}$.

(a) Show that $\frac{d^2y}{dx^2} = (4x^2 - 2)e^{-x^2}$. [3 marks]

(b) Find the exact coordinates of the points of inflection. [3 marks]

Answer .....

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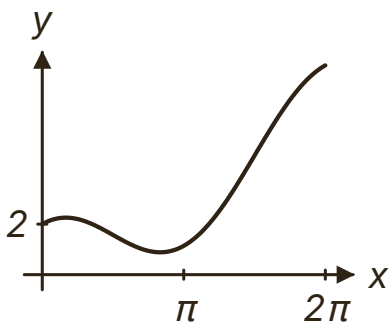
(c) State the set of values of x for which the curve is concave. [2 marks]

Answer .....

Total for D4: 8 marks

D5 The diagram shows the curve $y = x + 2\cos x$ for $0 \leqslant x \leqslant 2\pi$.

(a) Find the exact coordinates of the stationary points, and determine their nature. [5 marks]



Answer .....

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(b) Find the coordinates of the points of inflection. [2 marks]

Answer .....

(c) Find the set of values of x in $0 \leqslant x \leqslant 2\pi$ for which the curve is concave. [3 marks]

Answer .....

Total for D5: 10 marks

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D6 $f(x) = \dfrac{\ln x}{x}$, $x > 0$.

(a) Show that $f''(x) = \dfrac{2\ln x - 3}{x^3}$. [4 marks]

(b) Find the exact coordinates of the point of inflection, and the set of values of x for which f is convex. [4 marks]

Answer .....

Total for D6: 8 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The curve $y = x^3 + ax^2 + bx + c$ has a point of inflection at $(1, -2)$, and the gradient of the curve there is -3. Find a, b and c. Then show that the tangent to the curve at $(1, -2)$ meets the curve at no other point. [9 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Write down $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. The point of inflection gives one equation, the gradient a second, and the point on the curve a third.

Hint 2 $\frac{d^2y}{dx^2} = 6x + 2a$ is 0 at $x = 1$, so $a = -3$.

Hint 3 Set the curve equal to the tangent and factorise. You already know one root: it is a repeated one.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can find the interval on which a function is concave or convex using the second derivative.	○ ○ ○
I can find a point of inflection and check that the second derivative changes sign there.	○ ○ ○
I can explain why a second derivative of zero is not enough for a point of inflection.	○ ○ ○
I can classify stationary points, including a stationary point of inflection.	○ ○ ○

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