

Gradients of curves and differentiation from first principles

70 marks



Hints online, answers free
with an account:
mathswariz.uk/worksheets/a-level-differentiation-from-first-principles/

Date _____

A Do Now

Quick questions to start.

- | | |
|---|--|
| <p>A1 Needed today [2 marks]</p> <p>Expand and simplify
$(x + h)^3 - x^3$.</p> | <p>A2 Last lesson [2 marks]</p> <p>A particle moves in a straight line from $(2\mathbf{i} + 3\mathbf{j})$ m to $(8\mathbf{i} - 5\mathbf{j})$ m in 2 seconds. Find its speed.</p> |
|---|--|

Answer

Answer m s^{-1}

- | | |
|--|--|
| <p>A3 From chapter 5 [2 marks]</p> <p>Find the gradient of the straight line through $(-2, 7)$ and $(4, -11)$.</p> | <p>A4 From chapter 2 [2 marks]</p> <p>$f(x) = 2x^2 - 5x$. Find $f(3 + h)$, simplified.</p> |
|--|--|

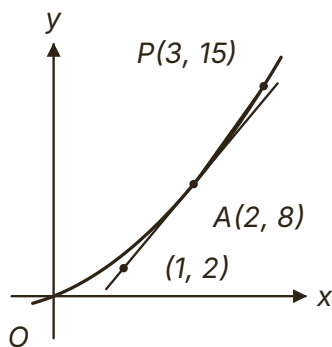
Answer

Answer $f(3 + h) = \dots\dots$

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

$A(2, 8)$ lies on $y = x^2 + 2x$. The tangent at A passes through $(1, 2)$. (a) Find the gradient of the curve at A . (b) Find the gradient of the chord AP when P has x -coordinate 3, then 2.1, then $2 + h$. (c) Comment.



Not drawn accurately

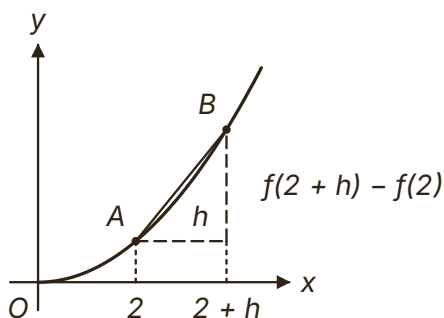
Hint The gradient of a curve at a point is the gradient of the tangent there. A chord's gradient is $\frac{y_2 - y_1}{x_2 - x_1}$.

B1 $A(3, -3)$ lies on $y = x^2 - 4x$. The tangent at A passes through $(5, 1)$. (a) Find the gradient of the curve at A . (b) Find the gradient of the chord AP when P has x -coordinate 4, then 3.1, then $3 + h$. [5 marks]

Answer (a) (b)

EXAMPLE 2

The point $A(2, 10)$ lies on the curve $y = f(x)$, where $f(x) = 3x^2 - x$. At A the curve has gradient g . (a) Show that $g = \lim_{h \rightarrow 0} (11 + 3h)$. (b) Deduce the value of g .



Not drawn accurately

Hint $g = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h}$. Expand $f(2 + h)$ carefully: square the whole bracket.

- B2** The point $A(2, 12)$ lies on the curve $y = x^3 + 2x$. The point B also lies on the curve and has x -coordinate $2 + h$. (a) Show that the gradient of AB is $h^2 + 6h + 14$. (b) Deduce the gradient of the curve at A . [5 marks]

Answer (b)

EXAMPLE 3

Prove, from first principles, that the derivative of $x^3 - 4x$ is $3x^2 - 4$.

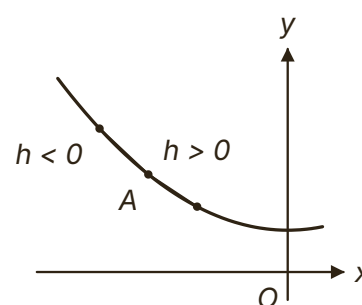
Hint $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$. Every term without an h cancels; then divide by h .

- B3** Prove, from first principles, that the derivative of $2x^2 + 7x - 1$ is $4x + 7$. [5 marks]

- C1** $A(4, 2)$ lies on $y = \sqrt{x}$. Find, to 4 d.p., the gradients of the chords from A to $P(4.41, 2.1)$ and from A to $Q(4.0401, 2.01)$. Hence estimate the gradient of the curve at A . [3 marks]

Answer AP : AQ : estimate

- C2** $A(-2, 7)$ lies on $y = x^2 + 3$. B is on the curve with x -coordinate $-2 + h$. Show that the gradient of AB is $h - 4$. Find this gradient when $h = 0.1$ and when $h = -0.1$, and state the gradient of the curve at A . [4 marks]



Not drawn accurately

Answer $h = 0.1$: $h = -0.1$: at A :

- C3** The points (x, y) and $(x + \delta x, y + \delta y)$ lie on the curve $y = 5 - 2x^2$. [4 marks]
Show that $\delta y = -4x \delta x - 2(\delta x)^2$, and hence find $\frac{dy}{dx}$.

Answer $\frac{dy}{dx} = \dots\dots$

D1 $f(x) = 2x^2 - 3x + 4.$

(a) Prove, from first principles, that $f'(x) = 4x - 3.$

[4 marks]

(b) The points P and Q on the curve $y = f(x)$ have x -coordinates 0 and 4. Find the gradient of the chord $PQ.$

[1 mark]

Answer

(c) Hence find the coordinates of the point R on the curve at which the gradient of the curve is equal to the gradient of $PQ.$

[3 marks]

Answer $R(\dots, \dots)$

Total for D1: 8 marks

D2

Kai finds the gradient of $y = x^3 + 5$ at $A(3, 32)$. His working is:
$$\frac{(3+h)^3 + 5 - 32}{h} = \frac{27 + h^3 - 27}{h} = h^2$$
, which tends to 0. He says the gradient of the curve at A is 0.

Kai is wrong.

(a) Describe the mistake in Kai's working. [2 marks]

(b) Hence find the correct gradient of the chord, in terms of h , and the gradient of the curve at A . [4 marks]

Answer chord at A :

Total for D2: 6 marks

D3 The curve C has equation $y = x^2 + kx$, where k is a constant. The point P on C has x -coordinate 3.

(a) Show that the gradient of the chord from P to the point on C with x -coordinate $3 + h$ is $6 + k + h$. [3 marks]

(b) The gradient of C at P is 11. Hence find the value of k .

[2 marks]

Answer $k = \dots\dots$

(c) Given that $\frac{dy}{dx} = 2x + k$, find the coordinates of the point on C at which the gradient is 0. [3 marks]

Answer $(\dots\dots, \dots\dots)$

Total for D3: 8 marks

D4 $f(x) = x^3 - 3x^2 + 4$.

(a) Prove, from first principles, that $f'(x) = 3x^2 - 6x$.

[5 marks]

- (b) Hence find the coordinates of the points on the curve $y = f(x)$ at [3 marks]
which the gradient is 9.

Answer (.....,.....) and (.....,.....)

Total for D4: 8 marks

E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

- E1** Prove, from first principles, that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$, for $x \neq 0$. [6 marks]

Extension hints

Cover these with paper. Uncover one at a time.

- E1** **Hint 1** Write the gradient of the chord as $\frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right)$.
- Hint 2** Put the two fractions over the common denominator $x(x+h)$: the numerator is $-h$.
- Hint 3** Cancel the h , then let $h \rightarrow 0$ in what is left.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can use chords to estimate a gradient. ☐ ☐ ☐

I can find a gradient from the definition. ☐ ☐ ☐

I can prove a derivative from first principles. ☐ ☐ ☐

I can find where the gradient takes a value. ☐ ☐ ☐