



Exponential functions and $y = e^x$

71 marks

Date _____

A Do Now

Quick questions to start.

- A1** Needed today [2 marks] Write $\left(\frac{1}{8}\right)^x \times 4^{x+1}$ as a single power of 2.
- A2** Needed today [2 marks] The curve $y = f(x)$ passes through (2, 5). Write down the point it moves to on $y = 3f(x)$, and on $y = f(x + 4)$.

Answer

Answer

- A3** Last lesson [2 marks] Find the area of the finite region enclosed by $y = x^2$ and $y = 3x$.
- A4** From chapter 12 [2 marks] Find the equation of the tangent to $y = x^2 - 3x$ at the point where $x = 4$.

Answer

Answer

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Sketch the curve $y = 2^{x+2} - 4$. Show where it meets the axes and the equation of its asymptote.

Hint Write $2^{x+2} = 4 \times 2^x$. Then it is $y = 2^x$ stretched by 4 upwards and moved down 4.

B1 Sketch the curve [4 marks]

$y = 4 - 3e^{-2x}$. Show where it meets the y -axis and the equation of its asymptote, and explain why $y < 4$ for every x .

EXAMPLE 2

The curve $y = ka^x$, where k and a are positive constants, passes through $(-1, 2)$ and $(2, 250)$. Find k and a .

Hint Write two equations and divide one by the other: k cancels.

B2 The curve $y = pq^x$, [4 marks]

where p and q are positive constants, passes through $(-1, 40)$ and $(2, 0.625)$. Find p and q .

Answer

EXAMPLE 3

$y = 3e^{2x} + e^{-x}$. Find $\frac{dy}{dx}$, and the equation of the tangent to the curve at the point where $x = 0$.

Hint The derivative of e^{kx} is ke^{kx} . The point is $(0, 3 + 1)$.

B3 $y = e^x(e^{2x} - 4)$. Find $\frac{dy}{dx}$, and the equation of the tangent to the curve at the point where $x = 0$. [4 marks]

Answer

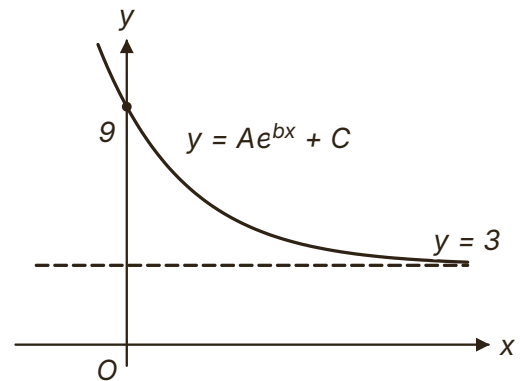
c Practice

A-Level

C1 On one set of axes, sketch $y = 4^x$, $y = 2^x$ and $y = \left(\frac{1}{4}\right)^x$. Show the point they all pass through. Describe the single transformation that maps $y = 4^x$ onto $y = \left(\frac{1}{4}\right)^x$, and explain why none of the three curves meets the x -axis. [4 marks]

- C2** The sketch shows part of the curve $y = Ae^{bx} + C$, where A , b and C are constants. The curve meets the y -axis at $(0, 9)$, and the line $y = 3$ is an asymptote. Find A and C , and state whether b is positive or negative. [3 marks]

Answer



- C3** Show that the gradient of the chord of $y = 3^x$ from x to $x + h$ is $3^x \times \frac{3^h - 1}{h}$. Hence explain why the gradient of $y = 3^x$ at any x is 3^x times its gradient at $x = 0$. Using $h = 0.0001$, estimate the gradient of $y = 3^x$ at $x = 2$, to 3 significant figures. [4 marks]

Answer

D1 The curve C_1 has equation $y = 4^x$ and the curve C_2 has equation $y = 3 \times 2^x + 4$.

- (a)** On one set of axes, sketch C_1 and C_2 . Show where each curve meets the y -axis and the equation of each asymptote. [4 marks]

- (b)** Show that the x -coordinates of the points where C_1 and C_2 meet satisfy $u^2 - 3u - 4 = 0$, where $u = 2^x$. Hence find the coordinates of the point where C_1 and C_2 meet, and explain why there is only one such point. [5 marks]

Answer

Total for D1: 9 marks

D2 $f(x) = e^{2x} - 4e^x + 2x$.

- (a)** Find $f'(x)$, and show that $f'(x) = 2(e^x - 1)^2$. [3 marks]

- (b) Hence show that the curve $y = f(x)$ has exactly one stationary point, and find its coordinates. Find $f''(x)$, and explain why $f''(0)$ does not show the nature of the stationary point. Determine its nature. [6 marks]

Answer

Total for D2: 9 marks

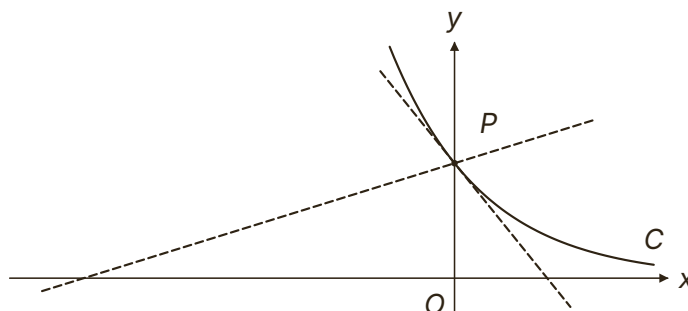
- D3** Jo writes: if $y = e^{3x+1} - 4x$, then $\frac{dy}{dx} = (3x + 1)e^{3x} - 4$ [7 marks]

Jo is wrong. Explain her mistake. Write e^{3x+1} in the form Ae^{3x} , where A is a constant, and hence find $\frac{dy}{dx}$ correctly. Show that the curve $y = e^{3x+1} - 4x$ is increasing for all $x \geq 0$, and find the exact gradient of the curve at the point where it crosses the y -axis.

Answer

D4 The curve C has equation $y = 4e^{-\frac{1}{2}x}$. The point P is where C meets the y -axis. The sketch shows C with its tangent and its normal at P .

- (a) Find $\frac{dy}{dx}$, and show [3 marks]
that the tangent to
 C at P has
equation
 $y = 4 - 2x$.



- (b) Find an equation of the normal to C at P . The tangent meets the x -axis at A and the normal meets the x -axis at B . Hence find the area of triangle ABP . [5 marks]

Answer

Total for D4: 8 marks

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 Prove that, for every positive integer n ,

[7 marks]

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$$

Use your result with $n = 5$ to give two bounds for e , to 3 decimal places.

Answer

Extension hints

Cover these with paper. Uncover one at a time.

E1 **Hint 1** Show first that $e^x > 1 + x$ for every $x \neq 0$. Then choose x well.

Hint 2 $g(x) = e^x - 1 - x$ has $g'(x) = e^x - 1$, so g is least at $x = 0$, where $g(0) = 0$.

Hint 3 Put $x = \frac{1}{n}$ and raise to the power n . Put $x = -\frac{1}{n+1}$: then $e^{-\frac{1}{n+1}} > \frac{n}{n+1}$, so $e^{\frac{1}{n+1}} < \frac{n+1}{n}$; raise to the power $n+1$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can sketch exponential curves, with their intercepts and asymptotes. ○ ○ ○

I can find k and a when a curve $y = ka^x$ passes through two points. ○ ○ ○

I can differentiate $e^{(kx)}$ and find tangents, normals and stationary points. ○ ○ ○