Integration by substitution

115 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-integration-by-substitution/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Given $u = 2x + 5$, write dx in terms of du.

Answer .....

A2 Needed today [3 marks]

Given $u = \sqrt{x - 1}$, write x in terms of u, and dx in terms of u and du.

Answer .....

A3 Common slip [2 marks]

An integral runs from $x = 1$ to $x = 5$. It is rewritten with $u = 2x - 1$. Write down the new limits for u.

Answer .....

A4 Needed today [2 marks]

Find $\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du$.

Answer .....

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B Example, then your turn

A-Level

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Use the substitution $u = 2x + 1$ to find $\int x\,(2x+1)^4\,dx$.

Hint From $u = 2x + 1$: $dx = \frac{1}{2}du$ and $x = \frac{u-1}{2}$. Every x must go, not only the bracket.

B1 Use the substitution $u = 3x - 1$ to find $\int x\,(3x-1)^5\,dx$. [5 marks]

Answer .....

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EXAMPLE 2

Use the substitution $u = \sqrt{x-1}$ to find the exact value of

$$\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx.$$

Hint $x = u^2 + 1$ and $dx = 2u\,du$. Change the limits: $x = 2$ gives $u = 1$, and $x = 10$ gives $u = 3$. Then stay in u.

B2 Use the substitution $u = \sqrt{x-3}$ to find the exact value of [6 marks]

$$\int_4^7 x\sqrt{x-3}\,dx.$$

Answer .....

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EXAMPLE 3

Use the substitution $u = e^x$ to find $\displaystyle\int \frac{e^{2x}}{e^x + 2}\,dx$.

Hint $du = e^x\,dx$, so $e^{2x}\,dx = e^x \times e^x\,dx = u\,du$. Then $\dfrac{u}{u+2} = 1 - \dfrac{2}{u+2}$.

B3 [5 marks]

Use the substitution $u = e^x$ to find the exact value of $\displaystyle\int_0^{\ln 3} \frac{e^{2x}}{e^x + 1}\,dx$.

Answer .....

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EXAMPLE 4

Use the substitution $u = 1 + \sin x$ to find the exact value of $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x}\, dx$.

Hint $du = \cos x\, dx$, which is already in the integral. The limits become $u = 1$ and $u = 2$.

B4 Use the substitution $u = 1 + \sin x$ to find the exact value of $\int_0^{\frac{\pi}{2}} \cos x \left(1 + \sin x\right)^3 dx$. [4 marks]

Answer .....

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EXAMPLE 5

Use the substitution $x = \sin\theta$ to find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$.

Hint $dx = \cos\theta\,d\theta$ and $\sqrt{1-\sin^2\theta} = \cos\theta$. When $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$.

B5 Use the substitution $x = 2\sin\theta$ to find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$. [4 marks]

Answer .....

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C1 [5 marks]

Zoe uses $u = x - 1$ on $\int_1^2 x\,(x-1)^4\,dx$. She writes $\int_1^2 (u+1)\,u^4\,du$ and gets $\frac{167}{10}$. Explain her mistake. Then find the correct value.

Answer .....

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C2 Ali uses $u = x^2 + 1$ on $\int x\left(x^2+1\right)^3\,dx$. He writes $\int xu^3\,du = \frac{xu^4}{4} + c$. Explain his mistake. Then find the integral. [4 marks]

Answer .....

C3 Use the substitution $u = \ln x$ to find the exact value of $\int_e^{e^2} \frac{1}{x\ln x}\,dx$. [4 marks]

Answer .....

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C4 Use the substitution $u = \cos x$ to find the exact value of $\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$. [5 marks]

Answer .....

C5 Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_1^9 \frac{1}{\sqrt{x}\,(1+\sqrt{x})} \, dx$. [4 marks]

Answer .....

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D1 This question is about $\displaystyle\int \frac{x}{\sqrt{2x+1}}\,dx$.

(a) Use the substitution $u^2 = 2x+1$ to show that $\displaystyle\int \frac{x}{\sqrt{2x+1}}\,dx = \frac{1}{3}(x-1)\sqrt{2x+1} + c$. [5 marks]

(b) Hence find the exact value of $\displaystyle\int_4^{12} \frac{x}{\sqrt{2x+1}}\,dx$. [2 marks]

Answer .....

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(c) Find the exact value of $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx$. [2 marks]

Answer .....

Total for D1: 9 marks

D2

Ben uses $u = 1 + 2\sin x$ on $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\cos x}{(1+2\sin x)^2}\,dx$. He writes

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} u^{-2}\,du.$$

Ben is wrong.

(a) Explain his mistake and write down the correct limits. [2 marks]

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(b) Find the exact value of the integral. [4 marks]

Answer .....

(c) Use the same substitution to find the exact value of [3 marks]

$$\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{(1+2\sin x)^2}\,dx.$$

Answer .....

Total for D2: 9 marks

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D3 Use the substitution $u^2 = e^x - 1$ to find the exact value of $\displaystyle\int_{\ln 2}^{\ln 5} \frac{e^{2x}}{\sqrt{e^x - 1}}\, dx$. [7 marks]

Answer .....

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D4 This question is about $\displaystyle\int_0^{\frac{3}{2}} \frac{x^2}{\sqrt{9-x^2}}\,dx$.

(a) Use the substitution $x = 3\sin\theta$ to show that the integral becomes $\displaystyle\int_0^{\frac{\pi}{6}} 9\sin^2\theta\,d\theta$. [3 marks]

(b) Hence find the exact value of the integral. Give your answer in the form $a\pi + b\sqrt{3}$. [6 marks]

Answer .....

Total for D4: 9 marks

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D5 This question is about substituting $u = \ln x$.

(a) Find $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$. [2 marks]

Answer .....

(b) Find the exact value of $\displaystyle\int_1^{e^2} \frac{(\ln x)^2}{x}\,dx$. [2 marks]

Answer .....

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(c) Find the exact value of $\displaystyle\int_{e}^{e^3} \frac{2}{x\left(\ln x\right)^2}\,dx$. [4 marks]

Answer .....

Total for D5: 8 marks

D6 This question is about the curve $y = x\sqrt{4-x^2}$ for $0 \leq x \leq 2$.

(a) Use the substitution $u = 4 - x^2$ to find the exact value of $\displaystyle\int_{0}^{2} x\sqrt{4-x^2}\,dx$. [4 marks]

Answer .....

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(b) Use the same substitution to find the exact value of [5 marks]

$$\int_0^2 x^3\sqrt{4-x^2}\,dx.$$

Answer .....

Total for D6: 9 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 Use the substitution $x = \tan\theta$ to find the exact value of $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx$. [9 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 $dx = \sec^2\theta\, d\theta$ and $1 + \tan^2\theta = \sec^2\theta$. The limits are $\theta = 0$ and $\theta = \dfrac{\pi}{4}$.

Hint 2 The integrand becomes $\dfrac{\sec^2\theta}{\sec^4\theta} = \cos^2\theta$.

Hint 3 Use $\cos^2\theta = \dfrac{1}{2} + \dfrac{1}{2}\cos 2\theta$, then integrate and substitute.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can choose a substitution, find dx in terms of du, and write the whole integral in terms of u.	○ ○ ○
I can change the limits of a definite integral so I never have to go back to x.	○ ○ ○
I can use a substitution such as $u^2 = 2x + 1$ or $u = e^x$ to integrate a function I cannot spot.	○ ○ ○
I can use $x = \sin\theta$ or $x = \tan\theta$ to find an exact value involving π.	○ ○ ○

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