Iteration

112 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-iteration/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

A sequence is given by $u_{n+1} = \frac{2}{u_n} + 1$, $u_1 = 1$.

Find u_2, u_3 and u_4.

Answer .....

A2 Needed today [2 marks]

Show that $x^3 - 4x - 1 = 0$ can be written as $x = \sqrt[3]{4x + 1}$.

A3 Common slip [2 marks]

Using $x_{n+1} = \sqrt{3x_n + 2}$ with $x_0 = 3$, find x_1 and x_2 to 4 decimal places. Use the full calculator value of x_1 to find x_2.

Answer .....

A4 Needed today [3 marks]

Given $\mathrm{f}(x) = x^3 - 3x - 1$, show that $\mathrm{f}(x) = 0$ has a root in $[1.8, 1.9]$.

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B Example, then your turn

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Show that $x^3 + 2x - 9 = 0$ can be written in the form $x = \sqrt[3]{9 - 2x}$.

Hint Keep x^3 on one side and move everything else across. Then take the cube root.

B1 Show that $x^3 - 6x + 2 = 0$ can be written in the form $x = \sqrt{6 - \frac{2}{x}}$, for $x > 0$. [3 marks]

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EXAMPLE 2

Use $x_{n+1} = \sqrt[3]{9 - 2x_n}$ with $x_0 = 1.5$ to find x_1, x_2, x_3 and x_4 to 4 decimal places.

Hint Type 1.5, press =, then type the formula using Ans and press = again and again. Write each value to 4 d.p. but never type the rounded value back in.

B2 [4 marks]

Use $x_{n+1} = \sqrt{6 - \dfrac{2}{x_n}}$ with $x_0 = 2$ to find x_1, x_2, x_3 and x_4 to 4 decimal places.

Answer .....

EXAMPLE 3

The iterates in the last example approach a root α of $x^3 + 2x - 9 = 0$. Show that $\alpha = 1.76$ to 2 decimal places.

Hint Iterates that look settled do not prove anything. Test the bounds 1.755 and 1.765 in $\mathrm{f}(x) = x^3 + 2x - 9$.

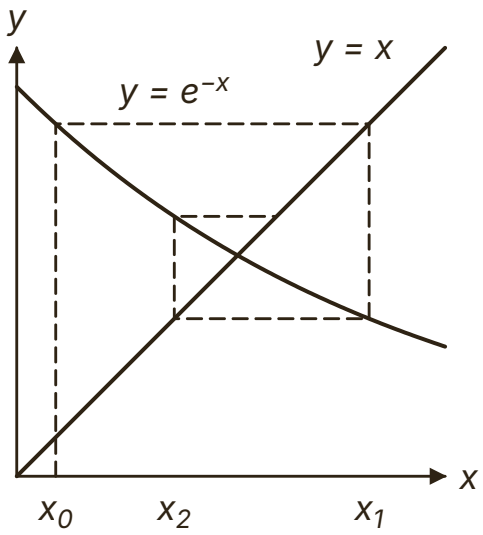
B3 [4 marks]

The iterates in B2 approach a root β of $x^3 - 6x + 2 = 0$. Show that $\beta = 2.262$ to 3 decimal places.

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EXAMPLE 4

The diagram shows $y = x$, $y = e^{-x}$ and the first steps of $x_{n+1} = e^{-x_n}$ from $x_0 = 0.1$. Find x_1, x_2, x_3. Say whether this is a staircase or a cobweb, and use $\mathrm{g}'(x)$ to explain why it converges.



Hint Up to the curve, across to $y = x$, up or down to the curve again. If the steps go round the root it is a cobweb. Then look at the size and the sign of $\mathrm{g}'(\alpha)$.

B4 For the iteration in B2, $\mathrm{g}(x) = \sqrt{6 - \dfrac{2}{x}}$. Say whether it gives a staircase or a cobweb and how you can tell from the iterates. Find $\mathrm{g}'(x)$ and show that $|\mathrm{g}'(\beta)| < 1$, where $\beta = 2.262$. [5 marks]

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EXAMPLE 5

The equation $x^2 - 5x + 3 = 0$ has roots 0.697 and 4.303. Use $x_{n+1} = \dfrac{x_n^2 + 3}{5}$ with $x_0 = 4$ to find x_1, x_2, x_3. Explain why this iteration cannot find the root 4.303.

Hint Watch where the iterates go. Then work out $\mathrm{g}'(x) = \dfrac{2x}{5}$ at each root.

B5 The equation $x^3 - 4x + 1 = 0$ has a root near 1.86. Two rearrangements are $x = \dfrac{x^3 + 1}{4}$ and $x = \sqrt[3]{4x - 1}$. Use $x_0 = 2$ in each to find x_1, x_2, x_3. Say which one finds the root near 1.86 and why. [5 marks]

Answer .....

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C1 Nia uses $x_{n+1} = 2 + \ln x_n$ with $x_0 = 3$. She finds x_3 and x_4 and says that because both round to 3.14, the root of $x - 2 - \ln x = 0$ is 3.14 to 2 decimal places. Find x_1 to x_4 and show that she is wrong. [6 marks]

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C2 Ben rearranges $x^2 - 3x - 5 = 0$ as $x = \sqrt{3x - 5}$. He uses it with $x_0 = 4$ and his calculator gives an error at x_4. Find his mistake. Write the correct rearrangement and use it with $x_0 = 4$ to find x_1 to x_4. [5 marks]

Answer .....

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C3 The equation $x^3 - 5x + 1 = 0$ has three roots. Use $x_{n+1} = \dfrac{x_n^3 + 1}{5}$ with $x_0 = 0.5$, and $x_{n+1} = \sqrt{5 - \dfrac{1}{x_n}}$ with $x_0 = 2$, to find x_1, x_2, x_3 for each. Explain why neither iteration can find the negative root, -2.330. [5 marks]

Answer .....

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C4 Kemi wants the root $\beta = 2.262$ of $x^3 - 6x + 2 = 0$. She uses $x_{n+1} = \dfrac{x_n^3 + 2}{6}$ with $x_0 = 2.3$. Find x_1, x_2, x_3 and use $\mathrm{g}'(x)$ to explain what happens. [5 marks]

D Exam-style questions

Edexcel exam style

D1 $\mathrm{f}(x) = \ln(x + 2) - x + 1$, $x > -2$. The equation $\mathrm{f}(x) = 0$ has a root α.

(a) Show that α lies in $[2.5, 2.6]$. [2 marks]

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(b) Show that $\mathrm{f}(x) = 0$ can be written as $x = 1 + \ln(x + 2)$. Use $x_{n+1} = 1 + \ln(x_n + 2)$ with $x_0 = 2$ to find x_1, x_2 and x_3 to 4 decimal places. [4 marks]

Answer .....

(c) Show that $\alpha = 2.505$ to 3 decimal places. [3 marks]

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(d) Use $g'(x)$ to explain why the iterates in (b) form a staircase. [2 marks]

Total for D1: 11 marks

D2

> Priya writes: "The roots of $x^2 - 6x + 4 = 0$ are $3 \pm \sqrt{5}$. If I start at $x_0 = 6$, the iteration $x_{n+1} = \dfrac{x_n^2 + 4}{6}$ will find the larger root, $3 + \sqrt{5}$."

Priya is wrong.

(a) Find x_1, x_2 and x_3 starting from $x_0 = 6$, and describe what happens. [3 marks]

Answer .....

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(b) Use $g'(x)$ to explain why this iteration can never find $3 + \sqrt{5}$. [2 marks]

(c) Starting from $x_0 = 1$, find x_1 and x_2 and state which root the iteration finds. [2 marks]

Answer .....

Total for D2: 7 marks

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D3 The equation $2\cos x = x^2$ has one positive root α. Use the iteration $x_{n+1} = \sqrt{2\cos x_n}$ with $x_0 = 1$ to find x_1 to x_4 to 4 decimal places, and describe how the iterates behave. Hence find α to 3 decimal places, showing that your answer is correct to this accuracy. Use radians. [7 marks]

Answer .....

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D4 The profit P, in thousands of pounds, of a pop-up shop t months after it opens is modelled by $P = 40 - 50e^{-0.4t} - 4t$. After about 0.75 months the shop makes a profit. Later the profit falls to zero again, at $t = T$.

(a) Show that T lies between 9 and 10. [2 marks]

(b) Show that $P = 0$ can be written as $t = 10 - 12.5e^{-0.4t}$. [2 marks]

(c) Use $t_{n+1} = 10 - 12.5e^{-0.4t_n}$ with $t_0 = 9$ to find t_1, t_2 and t_3 to 4 decimal places. [3 marks]

Answer .....

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(d) Find T to 2 decimal places, showing that your answer is correct. Give the time in months. [3 marks]

Answer .....

Total for D4: 10 marks

D5 A sequence is defined by $x_{n+1} = 4 - \sqrt{x_n + 5}$, $x_0 = 1$.

(a) Find x_1, x_2, x_3 and x_4 to 4 decimal places. [3 marks]

Answer .....

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(b) The sequence converges to α. Show that $\alpha^2 - 9\alpha + 11 = 0$ and hence find the exact value of α. [4 marks]

Answer .....

(c) Explain whether the iteration gives a staircase or a cobweb diagram. [2 marks]

Total for D5: 9 marks

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D6 The equation $e^x - 5x + 2 = 0$ has two roots, $\alpha \approx 0.88$ and $\beta \approx 2.19$.

(a) Show that the equation can be written as $x = \ln(5x - 2)$. [2 marks]

(b) Use $x_{n+1} = \ln(5x_n - 2)$ with $x_0 = 2$ to find x_1 to x_4 to 4 decimal places. State which root the iterates approach. [3 marks]

Answer .....

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(c) Starting at $x_0 = 0.8$, find x_1 and x_2. Explain why this iteration cannot be used to find α. [3 marks]

Total for D6: 8 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The equation $x^3 - 2x - 5 = 0$ has one real root α. An iteration is $x_{n+1} = x_n - \lambda\left(x_n^3 - 2x_n - 5\right)$, where λ is a positive constant. Find the range of values of λ for which iterates that start close to α converge to it, and the value of λ that gives the fastest convergence, to 3 significant figures. Then use $\lambda = 0.1$ and $x_0 = 2$ to find x_3 to 4 decimal places. [9 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Here $g(x) = x - \lambda\left(x^3 - 2x - 5\right)$, so $g'(x) = 1 - \lambda\left(3x^2 - 2\right)$.

Hint 2 Find α to 4 decimal places first, by a sign-change search. Then it converges when $-1 < g'(\alpha) < 1$.

Hint 3 It is fastest when $g'(\alpha) = 0$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can rearrange f(x) = 0 into the form x = g(x) and use the iterative formula with a starting value.	○ ○ ○
I can find iterates to a stated accuracy using the full calculator value each time.	○ ○ ○
I can draw a staircase or cobweb diagram and say which one an iteration gives.	○ ○ ○
I can use $\lvert g'(\alpha)\rvert < 1$ to explain why an iteration converges, diverges or finds another root.	○ ○ ○

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