



Mathematical proof

75 marks

Date _____

A Do Now

Quick questions to start.

- A1** [Last lesson] [2 marks] Use the factor theorem to show that $(x - 2)$ is a factor of $x^3 - 3x^2 - 4x + 12$.
- A2** [Needed today] [2 marks] Write $x^2 - 18x + 86$ in the form $(x - p)^2 + q$.

Answer

- A3** [From chapter 2] [2 marks] Find the discriminant of $3x^2 - 5x + 4$. Hence state the number of real roots of $3x^2 - 5x + 4 = 0$.
- A4** [From chapter 5] [2 marks] A line passes through $(-1, 4)$ and $(3, -4)$. Find the gradient of a line perpendicular to it.

Answer

Answer

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Prove that $(2x + 3)(x - 4)(x + 1)$
 $\equiv 2x^3 - 3x^2 - 17x - 12$.

Hint Start from the left-hand side. Multiply two of the brackets first, then the third. Do not work on both sides at once.

B1

Prove that $\left(2x + \frac{1}{x}\right)^3$ [3 marks]
 $\equiv 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$.

EXAMPLE 2

Prove that the sum of the squares of any three consecutive integers is always 2 more than a multiple of 3.

Hint Call the integers $n - 1$, n and $n + 1$, where n is an integer. Expand and collect.

B2

Prove that 1 more than the product of two consecutive even numbers is always the square of an odd number. [3 marks]

EXAMPLE 3

The equation $kx^2 + 4kx + 3 = 0$, where k is a constant, has no real roots. Prove that $0 \leq k < \frac{3}{4}$.

Hint No real roots means $b^2 - 4ac < 0$. That rule needs a quadratic, so look at $k = 0$ on its own.

B3 The equation [4 marks]

$px^2 + 6x - 2 = 0$, where p is a non-zero constant, has two distinct real roots. Prove that $p > -\frac{9}{2}$.

EXAMPLE 4

Prove that every square number is either a multiple of 3 or 1 more than a multiple of 3.

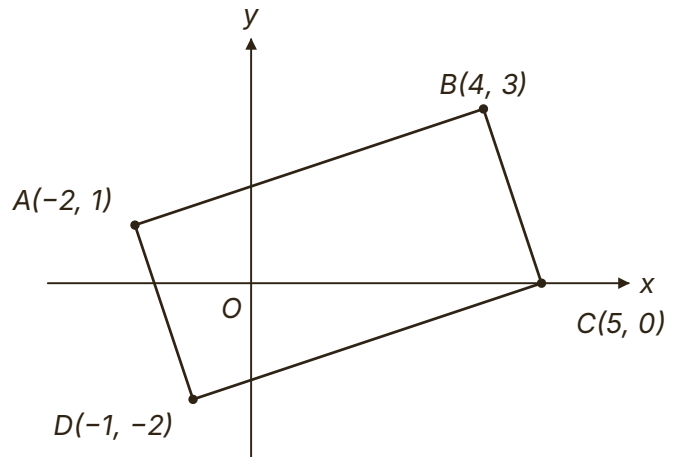
Hint Every integer n is $3k$, $3k + 1$ or $3k + 2$ for some integer k . Square each case.

B4 Prove that $n^2 + n + 1$ is [4 marks]
not divisible by 5 for any integer n .**c Practice**

A-Level

C1 Prove that $(2x - 1)^2 > 8x - 12$ for all real values of x . [3 marks]

- C2** Prove that the points $A(-2, 1)$, $B(4, 3)$, $C(5, 0)$ and $D(-1, -2)$ are the vertices of a rectangle, and that $ABCD$ is not a square. [4 marks]



Not drawn accurately

- C3** Prove that, when n is an odd integer, $n^2 - 1$ is a multiple of 8. [4 marks]

D Exam-style questions

Edexcel exam style

D1

Mia wants to prove that $x + \frac{9}{x} \geq 6$ for all positive real x . She writes:

$$x + \frac{9}{x} \geq 6 \Rightarrow x^2 + 9 \geq 6x \Rightarrow x^2 - 6x + 9 \geq 0 \Rightarrow (x - 3)^2 \geq 0, \text{ which is true. So } x + \frac{9}{x} \geq 6.$$

The statement is true but Mia's proof is not valid.

- (a)** Explain why Mia's argument is not a valid proof. [1 mark]

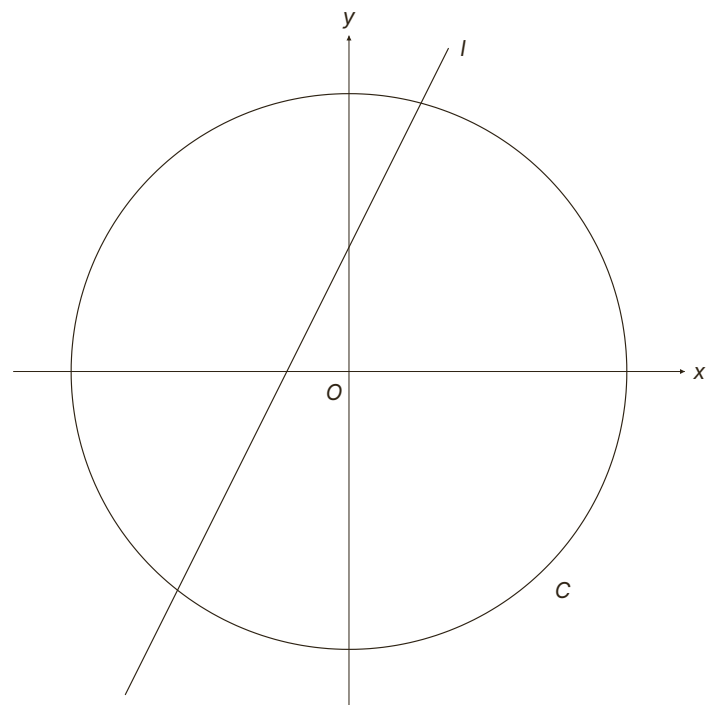
- (b) Write out a correct proof that $x + \frac{9}{x} \geq 6$ for all $x > 0$. State clearly where you use the fact that $x > 0$. [4 marks]

- (c) Use a counter-example to show that the inequality is not true for all real $x \neq 0$. [2 marks]

Total for D1: 7 marks

- D2** The circle C has equation $x^2 + y^2 = 125$. The line l has equation $y = 2x + c$, where c is a constant.

- (a) Show that the x -coordinates of any points where l meets C satisfy $5x^2 + 4cx + c^2 - 125 = 0$. [2 marks]



Not drawn accurately

- (b) Hence prove that l is a tangent to C when $c = 25$, and find the coordinates of the point where it touches C . [5 marks]

Answer (....., ...)

- (c) Find the set of values of c for which l meets C at two distinct points. [3 marks]

Answer

Total for D2: 10 marks

D3 In this question n is an integer.

- (a) Prove that $n^3 + 3n^2 + 2n$ is a multiple of 6. [4 marks]

(b) Hence prove that $n^3 + 3n^2 + 8n$ is a multiple of 6. [2 marks]

(c) Sam says: " $n^3 + 3n^2 + 2n$ is always a multiple of 12." Show that Sam is wrong. [1 mark]

(d) Prove that, when n is even, $n^3 + 3n^2 + 2n$ is a multiple of 24. [3 marks]

Total for D3: 10 marks

D4 $f(x) = x^2 + (k - 3)x + 3 - 2k$, where k is a constant.

(a) Show that the discriminant of $f(x)$ is $k^2 + 2k - 3$. [2 marks]

(b) Hence find the set of values of k for which $f(x) = 0$ has no real roots. [3 marks]

Answer

- (c) $g(x) = x^2 + (k - 3)x - 2k$. Prove that $g(x) = 0$ has two distinct real roots for every value of k . [4 marks]

Total for D4: 9 marks

E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

- E1** Prove that $n^5 - n$ is divisible by 30 for every integer n . [6 marks]

Extension hints

Cover these with paper. Uncover one at a time.

- E1** **Hint 1** $30 = 2 \times 3 \times 5$. Factorise $n^5 - n$ as far as you can and look for each of the three primes.
- Hint 2** $n^5 - n = n(n^4 - 1) = (n - 1)n(n + 1)(n^2 + 1)$. Three consecutive integers give the 2 and the 3.
- Hint 3** For 5, try the five cases $n = 5k, 5k \pm 1, 5k \pm 2$. The first three make $n, n - 1$ or $n + 1$ a multiple of 5. For $n = 5k \pm 2$, $n^2 + 1 = 25k^2 \pm 20k + 5$, a multiple of 5.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can prove an identity by working from one side to the other. ☐ ☐ ☐

I can prove a statement by deduction, by exhaustion or by completing the square. ☐ ☐ ☐

I can use the discriminant to prove a result about a constant. ☐ ☐ ☐

I can disprove a statement with one counter-example. ☐ ☐ ☐