Differentiating and integrating vectors

142 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-mechanics-year-2-differentiating-integrating-vectors/

Date

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A Do Now

Quick questions to start. Some check what you need today; some set a slip that the examiners' reports say costs marks.

A1 Needed today [3 marks]

Find the magnitude of the vector $(6\mathbf{i} - 8\mathbf{j})$, and the angle it makes with $\mathbf{i}$.

Answer .....

A2 Needed today [2 marks]

A particle moves on the x-axis with displacement $x = 2t^3 - 3\sin 2t$ metres at time t seconds. Find expressions for its velocity and its acceleration.

Answer .....

A3 Needed today [2 marks]

A particle has velocity $\mathbf{v} = \big((2t - 6)\mathbf{i} + (t + 1)\mathbf{j}\big)\ \text{m s}^{-1}$. Find the time when it moves parallel to $\mathbf{j}$, and the time when it moves parallel to $(\mathbf{i} + \mathbf{j})$.

Answer .....

A4 Common slip [2 marks]

A particle has acceleration $\mathbf{a} = (2\mathbf{i} + 6t\mathbf{j})\ \text{m s}^{-2}$, and $\mathbf{v} = (3\mathbf{i} - \mathbf{j})\ \text{m s}^{-1}$ when $t = 0$. Ali writes "$\mathbf{v} = 2t\mathbf{i} + 3t^2\mathbf{j} + 2$". Correct his answer.

Answer .....

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A5 Common slip [3 marks]

A particle has velocity $\mathbf{v} = \left((t+2)\mathbf{i} + (10-3t)\mathbf{j}\right)\ \text{m s}^{-1}$. Lee wants the time when it moves parallel to $(\mathbf{i} - 2\mathbf{j})$. He writes "$2(10-3t) = -(t+2)$" and gets $t = 4.4$. Correct his working.

Answer .....

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B Example, then your turn

A-Level

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

A particle P of mass $0.5\ \text{kg}$ moves in a horizontal plane. At time t seconds its position vector relative to a fixed point O is $\mathbf{r} = \left((t^3 - 6t)\mathbf{i} + 8\sqrt{t}\,\mathbf{j}\right)\ \text{m}$, $t > 0$. Find the speed of P when $t = 1$, and the magnitude of the resultant force on P when $t = 1$.

Hint Differentiate each component separately and keep $\mathbf{i}$ and $\mathbf{j}$ on every line. $\mathbf{F} = m\mathbf{a}$ is a vector equation.

B1 A particle P of mass $0.25\ \text{kg}$ moves in a horizontal plane. At time t seconds its position vector relative to O is $\mathbf{r} = \left((2t^3 - 9t)\mathbf{i} + 12\sqrt{t}\,\mathbf{j}\right)\ \text{m}$, $t > 0$. Find the speed of P when $t = 1$, and the magnitude of the resultant force on P when $t = 1$. [6 marks]

Answer .....

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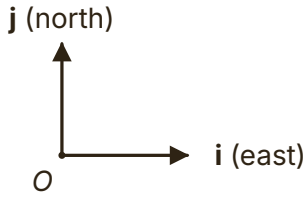
EXAMPLE 2

At time t seconds, $t \geq 0$, a particle P has position vector $\mathbf{r} = \left((t^3 - 12t)\mathbf{i} + (3t^2 + 12t)\mathbf{j}\right)$ m relative to O, where $\mathbf{i}$ is due east and $\mathbf{j}$ is due north. Find the position vector of P when it is moving due north. Find the time when P is moving north-east.



Hint Moving due north: the $\mathbf{i}$-component of $\mathbf{v}$ is zero and the $\mathbf{j}$-component is positive. Moving north-east: the two components of $\mathbf{v}$ are equal and positive.

B2 At time t seconds, $t \geq 0$, a particle P has position vector $\mathbf{r} = \left((t^3 - 27t)\mathbf{i} + (3t^2 + 18t)\mathbf{j}\right)$ m relative to O, where $\mathbf{i}$ is due east and $\mathbf{j}$ is due north. Find the position vector of P when it is moving due north, and the time when it is moving north-east. [7 marks]

Answer .....

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EXAMPLE 3

A particle P moves in a plane with acceleration $\mathbf{a} = (2\mathbf{i} - 6t\mathbf{j})\ \text{m s}^{-2}$ at time t seconds. When $t = 1$, $\mathbf{v} = (5\mathbf{i} - 2\mathbf{j})\ \text{m s}^{-1}$ and $\mathbf{r} = (3\mathbf{i} + 4\mathbf{j})\ \text{m}$ relative to O. Find $\mathbf{v}$ and $\mathbf{r}$ in terms of t, and the distance of P from O when $t = 0$. Given that $\mathbf{i}$ is due east and $\mathbf{j}$ is due north, find the bearing on which P is moving when $t = 2$.

Hint Integrate with a vector constant $\mathbf{c}$, then use the condition at $t = 1$, not at $t = 0$. Draw $\mathbf{v}(2)$ to read off the bearing.

B3 [7 marks]

A particle P moves in a plane with acceleration $\mathbf{a} = (6t\mathbf{i} - 4\mathbf{j})\ \text{m s}^{-2}$ at time t seconds. When $t = 2$, $\mathbf{v} = (10\mathbf{i} - 5\mathbf{j})\ \text{m s}^{-1}$ and $\mathbf{r} = (7\mathbf{i} + \mathbf{j})\ \text{m}$ relative to O. Find $\mathbf{v}$ and $\mathbf{r}$ in terms of t, and the distance of P from O when $t = 0$. Given that $\mathbf{i}$ is due east and $\mathbf{j}$ is due north, find the bearing on which P is moving when $t = 1$.

Answer .....

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EXAMPLE 4

At time t seconds a particle P has position vector $\mathbf{r} = \big((t^2 + 4t)\mathbf{i} + (kt^2 - 2t^3)\mathbf{j}\big)$ m, where k is a constant. When $t = 2$ the speed of P is $10\ \text{m s}^{-1}$. Find the two possible values of k, and the magnitude of the acceleration of P when $t = 2$ for each value.

Hint Find $\mathbf{v}$ at $t = 2$ in terms of k, then use speed^2 = ($\mathbf{i}$-component)^2 + ($\mathbf{j}$-component)^2.

B4 [6 marks]

At time t seconds a particle P has position vector $\mathbf{r} = \big((t^2 - 6t)\mathbf{i} + (kt^2 - t^3)\mathbf{j}\big)$ m, where k is a constant. When $t = 3$ the speed of P is $15\ \text{m s}^{-1}$. Find the two possible values of k. For each value, find the exact magnitude of the acceleration of P when $t = 1$.

Answer .....

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EXAMPLE 5

Two particles P and Q move in a plane. P has velocity $\left((6t-4)\mathbf{i}+3\mathbf{j}\right)\ \mathrm{m\ s^{-1}}$ at time t seconds and is at the point with position vector $(\mathbf{i}+2\mathbf{j})\ \mathrm{m}$ when $t=0$. Q moves with constant velocity $(5\mathbf{i}+2\mathbf{j})\ \mathrm{m\ s^{-1}}$ and is at $(\mathbf{i}+5\mathbf{j})\ \mathrm{m}$ when $t=0$. Show that P and Q collide, and find where. Find the distance between them when $t=1$.

Hint Write $\mathbf{r}_Q=\mathbf{r}_0+\mathbf{v}t$. They collide only if BOTH components agree at the SAME time.

B5 [7 marks]

Particle P has velocity $\left((4t-3)\mathbf{i}+2\mathbf{j}\right)\ \mathrm{m\ s^{-1}}$ at time t seconds and is at $(2\mathbf{i}-\mathbf{j})\ \mathrm{m}$ when $t=0$. Particle Q moves with constant velocity $(5\mathbf{i}+k\mathbf{j})\ \mathrm{m\ s^{-1}}$ and is at $(-4\mathbf{i}+7\mathbf{j})\ \mathrm{m}$ when $t=0$. Given that P and Q collide, find the two possible values of k, and the position vector of the collision for each.

Answer .....

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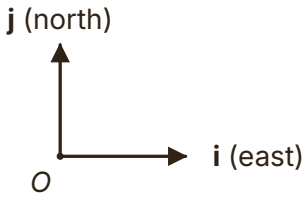
C1 At time t seconds a particle has position vector $\mathbf{r} = \big((t^2 - 3t + 4)\mathbf{i} + (t+1)\mathbf{j}\big)$ m relative to O, where $\mathbf{i}$ is due east and $\mathbf{j}$ is due north. Ava says "The particle is north-east of O when its velocity has equal components, so when $2t - 3 = 1$." Explain Ava's mistake. Find the times when the particle is north-east of O, and the time when it is moving north-east. [5 marks]



Answer .....

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C2 A particle has velocity $\mathbf{v} = \left((t^2 - 4t)\mathbf{i} + (6 - t)\mathbf{j}\right)\ \text{m s}^{-1}$ at time t seconds, $t \geq 0$. Ben wants the times when it moves parallel to $\mathbf{j}$. He writes "$6 - t = 0$, so $t = 6$." Explain Ben's mistake. Find the times when the particle moves parallel to $\mathbf{j}$, and its acceleration at each time. [5 marks]

Answer .....

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C3 A particle starts at O and has position vector $\mathbf{r} = \left((t^3 - 2t)\mathbf{i} + 4t^2\mathbf{j}\right)$ m at time t seconds. Mia finds its velocity when $t = 2$ by dividing $\mathbf{r}(2)$ by 2, and gets $(2\mathbf{i} + 8\mathbf{j})$ m s^{-1}. Explain why Mia's method is not valid, and find the correct speed when $t = 2$. [4 marks]

Answer .....

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C4 A particle has position vector $\mathbf{r} = \left((3t^2 - 4t)\mathbf{i} + (5 - 2t^2)\mathbf{j}\right)$ m at time t seconds. Show that its acceleration is constant. Pete says "The magnitude of the acceleration is $\sqrt{6^2 - 4^2} = 4.47\ \text{m s}^{-2}$." Find the correct magnitude, and the angle between the acceleration and $\mathbf{i}$. [6 marks]

Answer .....

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C5 A particle P moves in a plane with velocity $\mathbf{v} = \big((3t^2 - 4)\mathbf{i} + (2t + 1)\mathbf{j}\big)\ \text{m s}^{-1}$ at time t seconds. When $t = 2$, P is at the point with position vector $(5\mathbf{i} + 3\mathbf{j})\ \text{m}$ relative to O. Tom writes "$\mathbf{r} = (t^3 - 4t)\mathbf{i} + (t^2 + t)\mathbf{j} + (5\mathbf{i} + 3\mathbf{j})$". Explain Tom's mistake, find $\mathbf{r}$ in terms of t, and find the distance of P from O when $t = 1$. [5 marks]

Answer .....

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D Exam-style questions

Edexcel exam style

D1 A particle P of mass $0.5\ \text{kg}$ moves in a horizontal plane. At time t seconds, $t \geq 0$, its velocity is $\mathbf{v} = \left((6 - 6\text{e}^{-t})\mathbf{i} + 8\text{e}^{-t}\mathbf{j}\right)\ \text{m s}^{-1}$. P is at the fixed point O when $t = 0$.

(a) Find the initial speed of P. [2 marks]

Answer .....

(b) Find the magnitude of the resultant force on P when $t = \ln 2$. [4 marks]

Answer .....

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(c) Show that the resultant force on P always acts in the same direction, and find the angle between this direction and $\mathbf{i}$. [3 marks]

Answer .....

(d) Find the position vector of P when $t = \ln 2$, giving your answer in exact form. [4 marks]

Answer .....

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(e) Describe the motion of P when t is large. [2 marks]

Total for D1: 15 marks

D2

> Jo writes: "For P to move parallel to $(2\mathbf{i} + \mathbf{j})$ I need $2(t^2 + 8) = 3t$. This gives $2t^2 - 3t + 16 = 0$, which has no real solutions, so P never moves parallel to $(2\mathbf{i} + \mathbf{j})$."

A particle P of mass $2\ \text{kg}$ starts from O at $t = 0$ and moves in a horizontal plane with velocity $\mathbf{v} = \left((t^2 + 8)\mathbf{i} + 3t\mathbf{j}\right)\ \text{m s}^{-1}$ at time t seconds.

(a) Explain the mistake in Jo's working. [2 marks]

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(b) Find the times when P moves parallel to $(2\mathbf{i} + \mathbf{j})$. [4 marks]

Answer .....

(c) Find the distance between the positions of P at these two times. [5 marks]

Answer .....

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(d) Find the magnitude of the resultant force on P at the earlier of the two times. [3 marks]

Answer .....

Total for D2: 14 marks

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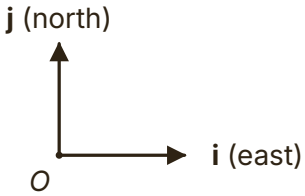
D3 A particle P moves in a plane, where $\mathbf{i}$ is due east and $\mathbf{j}$ is due north. At time t seconds its acceleration is $\big((12t-6)\mathbf{i}+4\mathbf{j}\big)\ \mathrm{m\,s^{-2}}$. When $t=0$, P is at the fixed point O and its velocity is $(-36\mathbf{i}-8\mathbf{j})\ \mathrm{m\,s^{-1}}$. Find the distance of P from O at the instant when P is moving due north. [8 marks]



Answer .....

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D4 Two particles P and Q move in a plane. At time t seconds, $t \geq 0$, their position vectors relative to O are $\mathbf{r}_P = \left((2t^2 + 1)\mathbf{i} + (4t - 1)\mathbf{j}\right)$ m and $\mathbf{r}_Q = \left((t + 7)\mathbf{i} + (t^2 + 3)\mathbf{j}\right)$ m.

(a) Find the velocities of P and Q in terms of t. [3 marks]

Answer .....

(b) Find the time when P and Q are moving parallel to each other. [3 marks]

Answer .....

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(c) Show that P and Q collide, and find the position vector of the point where they collide. [4 marks]

Answer .....

(d) Find the distance between P and Q when $t = 0$. [2 marks]

Answer .....

Total for D4: 12 marks

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D5 A particle P moves in a plane. At time t seconds its acceleration is $\mathbf{a} = \left((2t-6)\mathbf{i} + 8\cos 2t\,\mathbf{j}\right)\ \mathrm{m\ s^{-2}}$. When $t = \dfrac{\pi}{2}$, P is instantaneously at rest.

(a) Find $\mathbf{v}$ in terms of t and π. [4 marks]

Answer .....

(b) Find the speed of P when $t = \dfrac{3\pi}{4}$. [3 marks]

Answer .....

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(c) Show that the $\mathbf{i}$-component of $\mathbf{v}$ is zero again at $t = 6 - \frac{\pi}{2}$, but that P is not at rest at this time. [3 marks]

Answer .....

(d) Find the time t, with $0 < t < 4$, at which P is moving parallel to $\mathbf{i}$. [3 marks]

Answer .....

Total for D5: 13 marks

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No route is given. The hints are at the end of the sheet. Use one at a time.

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E1 A particle P moves in a horizontal plane. At time t seconds its position vector relative to O is $\mathbf{r} = \left(4\sin\omega t\,\mathbf{i} + 3\cos\omega t\,\mathbf{j}\right)\ \mathrm{m}$, where ω is a positive constant. Find, in terms of ω, the least and greatest speeds of P. Given that the greatest speed of P is $12\ \mathrm{m\ s^{-1}}$, find the distance of P from O at the first time $t > 0$ when P is moving parallel to $(\mathbf{i} - \mathbf{j})$. [10 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Square and add the components of $\mathbf{v}$, then use $\sin^2 = 1 - \cos^2$ to write the speed² in terms of $\cos^2 \omega t$ only.

Hint 2 Moving parallel to $\mathbf{i} - \mathbf{j}$ means $\mathbf{v}$ is a POSITIVE multiple of $\mathbf{i} - \mathbf{j}$: the $\mathbf{i}$-component equals minus the $\mathbf{j}$-component, and the $\mathbf{i}$-component is positive.

Hint 3 $\tan 3t = \frac{4}{3}$ gives a 3-4-5 triangle, so $\sin 3t$ and $\cos 3t$ are exact.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can differentiate and integrate position, velocity and acceleration vectors given in terms of t.	○ ○ ○
I can find vector constants of integration from a condition at any time.	○ ○ ○
I can find when a particle moves parallel to i, to j or to a given vector, and when it is due north of a point.	○ ○ ○
I can decide whether two particles collide and find the distance between them.	○ ○ ○

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