Modelling with differential equations

117 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-modelling-with-differential-equations/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

The rate of increase of a quantity N at time t is proportional to N. Write this as a differential equation.

Answer .....

A2 Last lesson [2 marks]

Given that $y = 5$ when $x = 0$, solve $\frac{dy}{dx} = -2y$, $y > 0$.

Answer .....

A3 Needed today [2 marks]

Solve $e^{-0.2t} = 0.25$, giving the exact value of t.

Answer .....

A4 From chapter 1 [2 marks]

Express $\frac{1}{P(10 - P)}$ in partial fractions.

Answer .....

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B Example, then your turn

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

A cup of tea at $90\,^\circ\mathrm{C}$ is left in a room at $20\,^\circ\mathrm{C}$. Its temperature $\theta\,^\circ\mathrm{C}$ falls at a rate proportional to the difference between θ and 20. After 5 minutes it is $62\,^\circ\mathrm{C}$. How long does it take to cool to $40\,^\circ\mathrm{C}$?

Hint Falling means a minus sign: $\frac{d\theta}{dt} = -k(\theta - 20)$ with $k > 0$. Use the start value to find the constant of integration, then the 5-minute value to find k.

B1 A metal bar at $300\,^\circ\mathrm{C}$ is left to cool in a room at $25\,^\circ\mathrm{C}$. Its temperature falls at a rate proportional to the difference between its temperature and $25\,^\circ\mathrm{C}$. After 10 minutes it is $135\,^\circ\mathrm{C}$. Find its temperature after 20 minutes. [5 marks]

Answer .....

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EXAMPLE 2

A radioactive substance decays at a rate proportional to its mass m grams. Its half-life is 8 days and there are 50 grams at the start. How long until 5 grams are left?

Hint Solve $\frac{dm}{dt} = -km$ to get $m = 50e^{-kt}$. Half-life 8 days means $e^{-8k} = \frac{1}{2}$.

B2 A population of bacteria grows at a rate proportional to its size. At the start there are 400 bacteria, and the population doubles every 3 hours. Find the exact time when the population reaches $10\,000$, and give it to 3 significant figures. [5 marks]

Answer .....

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EXAMPLE 3

Water leaks from a tank so that the depth h cm after t minutes satisfies $\dfrac{dh}{dt} = -k\sqrt{h}$. At the start $h = 25$, and after 2 minutes $h = 16$. When is the tank empty?

Hint $\displaystyle\int h^{-1/2}\,dh = 2\sqrt{h}$. You need two conditions: one for c and one for k.

B3 [6 marks]

An ice cube melts so that its volume $V\ \text{cm}^3$ decreases at a rate proportional to $V^{2/3}$. At the start $V = 1000$, and after 1 hour $V = 512$. Show that $V^{1/3} = 10 - 2t$, where t is the time in hours, and find when the cube has melted.

Answer .....

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EXAMPLE 4

A population P (in thousands) satisfies $\dfrac{dP}{dt} = \dfrac{1}{20}P(10 - P)$, and $P = 2$ when $t = 0$. Find P in terms of t and the value P settles towards.

Hint Use $\dfrac{1}{P(10-P)} = \dfrac{1}{10}\left(\dfrac{1}{P} + \dfrac{1}{10-P}\right)$, then combine the logs into $\ln\dfrac{P}{10-P}$.

B4 [6 marks]

The number of people N (in thousands) who have heard a rumour after t days satisfies $\dfrac{dN}{dt} = \dfrac{1}{12}N(6 - N)$, and $N = 1$ when $t = 0$. Show that $N = \dfrac{6}{1 + 5e^{-t/2}}$, and find the exact time when 3 thousand people have heard it.

Answer .....

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C1 An oil slick is a circle of radius r cm. The radius increases at a rate inversely proportional to the radius. At the start $r = 2$, and after 3 hours $r = 4$. Form and solve a differential equation, and find the radius after 10 hours, giving an exact answer. [5 marks]

Answer .....

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C2 The concentration C of a drug in the blood falls at a rate proportional to C. At the start $C = 10$, and after 2 hours $C = 6$. Find C after 5 hours, and the time when C first falls to 1. Give both answers to 3 significant figures. [5 marks]

Answer .....

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C3 The number of fish P in a lake is modelled by $\frac{dP}{dt} = 0.02P - 30$, where t is in years and 30 fish are caught each year. Explain what the term $0.02P$ means. Given that $P = 2000$ when $t = 0$, solve the equation and find when P reaches 3000, to 3 significant figures. Say what the model predicts if instead $P = 1000$ when $t = 0$. [6 marks]

Answer .....

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C4 A student learns new words. The percentage N of a list learnt after t days increases at a rate proportional to $100 - N$. At the start $N = 0$, and after 5 days $N = 36$. Form and solve a differential equation, find N after 10 days, and state the value N settles towards. [6 marks]

Answer .....

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D1 A tank is a cylinder with a horizontal base of area $2\,\text{m}^2$. Water leaks out through a hole in the base. When the depth of water is h metres, the volume $V\,\text{m}^3$ decreases so that $\frac{dV}{dt} = -0.4\sqrt{h}$, where t is in minutes. At the start the depth is 4 m.

(a) Show that $\frac{dh}{dt} = -0.2\sqrt{h}$. [2 marks]

(b) Find h in terms of t. [4 marks]

Answer .....

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(c) Find the time when the tank is empty, and the exact time when the depth is 2 m. [3 marks]

Answer .....

Total for D1: 9 marks

D2

Leo models a cup of coffee cooling in a room at $20\,^{\circ}\mathrm{C}$. At the start it is $80\,^{\circ}\mathrm{C}$, and after 5 minutes it is $60\,^{\circ}\mathrm{C}$. He writes: "The rate of cooling is proportional to the temperature, so $\frac{d\theta}{dt} = -k\theta$ and $\theta = 80e^{-kt}$. This means that after a long time the coffee cools to $0\,^{\circ}\mathrm{C}$."

Leo makes two mistakes.

(a) Explain the mistake in his differential equation. [2 marks]

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(b) Explain why his conclusion about the long-term temperature is wrong. [2 marks]

(c) Solve the correct differential equation and find the exact value of e^{-5k}. [4 marks]

Answer .....

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(d) Find the temperature after 20 minutes from the correct model and from Leo's model, to 3 significant figures. [2 marks]

Answer .....

Total for D2: 10 marks

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D3 The number of birds N on an island after t years satisfies $\frac{dN}{dt} = \frac{N(800 - N)}{4000}$. At the start there are 100 birds. Find N in terms of t, and find the time when there are 400 birds, giving an exact answer and its value to 3 significant figures. [9 marks]

Answer .....

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D4 A patient is given a drug through a drip at a constant rate of 6 mg per hour. The body removes the drug at a rate of $0.5x$ mg per hour, where x mg is the amount of drug in the body after t hours. At the start there is no drug in the body.

(a) Write down a differential equation for x. [2 marks]

Answer .....

(b) Solve your equation to find x in terms of t. [4 marks]

Answer .....

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(c) Find the exact time when there are 10 mg of the drug in the body. [2 marks]

Answer .....

(d) State the amount the model settles towards. A doctor wants this amount to be 15 mg. Find the drip rate needed. [2 marks]

Answer .....

Total for D4: 10 marks

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D5 A population of rabbits P after t years is modelled by $\frac{dP}{dt} = 0.3P\cos(0.5t)$. At the start there are 500 rabbits.

(a) Find P in terms of t. [4 marks]

Answer .....

(b) Find the greatest number of rabbits predicted, and the first time it is reached, to 3 significant figures. [3 marks]

Answer .....

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(c) Find the least number of rabbits predicted, and give one reason why the model may not be suitable over many years. [2 marks]

Total for D5: 9 marks

D6 25 g of salt is put in water. After 4 minutes, 16 g has not yet dissolved. Two models are suggested for the mass x g not yet dissolved after t minutes.
Model 1: $\frac{dx}{dt} = -kx$. Model 2: $\frac{dx}{dt} = -k\sqrt{x}$.

(a) Use Model 1 to find x after 10 minutes, to 3 significant figures. [3 marks]

Answer .....

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(b) Use Model 2 to find x after 10 minutes. [3 marks]

Answer .....

(c) After 20 minutes, 2.6 g has not dissolved. Which model fits better? Give a reason. [2 marks]

Total for D6: 8 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 A mothball is a sphere. It evaporates so that its volume decreases at a rate proportional to its surface area. At the start its radius is 3 mm, and after 2 weeks its volume is half of what it was at the start. Assuming it stays a sphere, show that its radius decreases at a constant rate, and find how long it takes to disappear completely. Give an exact answer and its value to 3 significant figures. [10 marks]

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Write $\frac{dV}{dt} = -k \times 4\pi r^2$, and use the chain rule with $V = \frac{4}{3}\pi r^3$.

Hint 2 You should find $\frac{dr}{dt} = -k$, so $r = 3 - kt$.

Hint 3 Half the volume means r^3 is half of 27, so $r = 3 \times 2^{-1/3}$ when $t = 2$. Then set $r = 0$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can turn words such as proportional to, inversely proportional to and proportional to the difference into a differential equation with the right sign.	○ ○ ○
I can solve the equation and use two pieces of information to find both the constant of integration and the constant of proportion.	○ ○ ○
I can use the solution to predict a value or a time, and find the value the model settles towards.	○ ○ ○
I can say what a term in a model means, compare two models with data, and give a reason a model may not hold.	○ ○ ○

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