

Modelling with differentiation

72 marks



Hints online, answers free
with an account:
mathswariz.uk/worksheets/a-level-modelling-with-differentiation/

Date _____

A Do Now

Quick questions to start.

- A1** [2 marks] **Needed today**
A cuboid has a square base of side x cm and a volume of 96 cm^3 . Show that its surface area is $2x^2 + \frac{384}{x} \text{ cm}^2$.
- A2** [2 marks] **Last lesson**
Find the coordinates of the stationary point of $y = 2x + \frac{18}{x}$, $x > 0$.

Answer (....., ...)

Answer

- A3** [2 marks] **From chapter 2**
Find the greatest value of $40t - 5t^2$, and the value of t at which it happens.
- A4** [2 marks] **From chapter 1**
Write $\frac{3x^2 - 8}{2\sqrt{x}}$ in the form $ax^p + bx^q$.

Answer when $t = \dots\dots$

Answer

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

A reservoir holds $V = 50 + 6t^2 - \frac{1}{3}t^3$ million litres t weeks into a wet season, $0 \leq t \leq 15$. Find $\frac{dV}{dt}$ at $t = 2$ and say what it means. When is V rising fastest, and how fast?

Hint The rate $\frac{dV}{dt}$ is greatest where its own derivative is zero.

B1 A culture holds [5 marks]

$N = 400 + 30t^2 - t^3$
bacteria t hours after the start, $0 \leq t \leq 25$. Find the rate of increase of N at $t = 4$, with its units. When is N increasing fastest, and how fast?

EXAMPLE 2

A closed box has a square base of side x cm and surface area 216 cm^2 . Show that its volume is $V = 54x - \frac{1}{2}x^3 \text{ cm}^3$. Find the greatest volume; show it is a maximum.

Hint Write the height in terms of x first.

B2 An open box has a [5 marks]

square base of side x cm and volume 32 cm^3 . Show that its area is $A = x^2 + \frac{128}{x} \text{ cm}^2$. Find the least area; show it is a minimum.

EXAMPLE 3

A closed tin is a cylinder of radius r cm and volume 250π cm³. Show that its surface area is $S = 2\pi r^2 + \frac{500\pi}{r}$ cm². Find the exact least surface area, and show that it is a minimum.

Hint Write the height in terms of r . Keep π as π .

B3 A cylinder with a base [5 marks]

but no lid has radius r cm and surface area 75π cm². Show that its volume is

$$V = \frac{\pi}{2}(75r - r^3) \text{ cm}^3.$$

Find the exact greatest volume, and show that it is a maximum.

c Practice

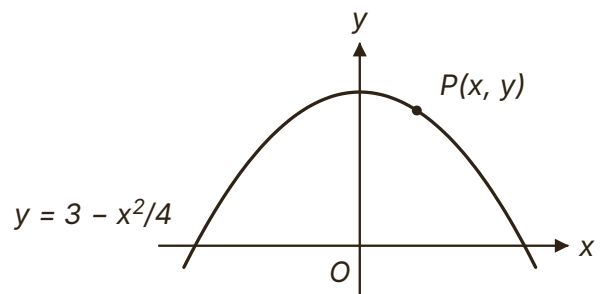
A-Level

- C1** Water is poured into a conical funnel. When the depth of water is h [2 marks]
cm, the volume of water is $V = \frac{\pi}{12}h^3$ cm³. Find $\frac{dV}{dh}$ when $h = 6$, and
explain what your answer means.

Answer $\frac{dV}{dh} = \dots\dots$

- C2** A lorry makes a journey of 240 km at a steady speed of v km/h. The driver is paid £20 per hour, and the running costs are £ $\frac{v}{80}$ for each kilometre. Show that the total cost of the journey, £ C , is $C = \frac{4800}{v} + 3v$. Find the speed that makes the cost least and the least cost, and show that it is a minimum. [5 marks]

- C3** The point $P(x, y)$ lies on the curve $y = 3 - \frac{1}{4}x^2$, and O is the origin. Show that $OP^2 = \frac{1}{16}x^4 - \frac{1}{2}x^2 + 9$. Hence find the shortest distance from O to the curve. [5 marks]

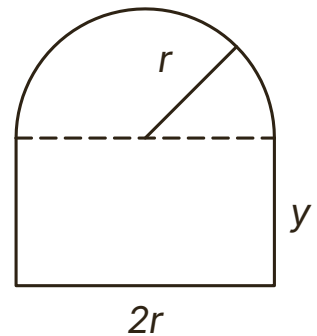


Not drawn accurately

Answer shortest distance = ...

- D1** A window is a rectangle with a semicircle on top, as shown. The rectangle is $2r$ m wide and y m tall, and the semicircle has radius r m. The perimeter of the whole window is 10 m.

- (a) Show that the area of the window, A m², [3 marks]
is $A = 10r - 2r^2 - \frac{1}{2}\pi r^2$.



Not drawn accurately

- (b) Use calculus to find the exact value of r for which A is greatest, [5 marks]
and show that it is a maximum.

Answer $r = \dots\dots$

Total for D1: 8 marks

D2

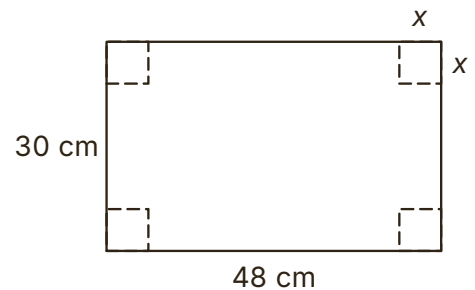
A box with no lid is made from a rectangular sheet of card 48 cm by 30 cm, by cutting a square of side x cm from each corner and folding up the sides. Sam finds that $\frac{dV}{dx} = 0$ when $x = 6$ and when $x = 20$, and writes: "The box is biggest when $x = 20$, because 20 is the larger value."

Sam's conclusion is wrong.

- (a) Show that the volume of the box, V cm³, is [2 marks]

V cm³, is

$$V = 4x^3 - 156x^2 + 1440x.$$



Not drawn accurately

- (b) Explain why $x = 20$ is not possible. Find the greatest volume of the box, and show that it is a maximum. [6 marks]

Answer greatest volume = ... cm³

Total for D2: 8 marks

- D3** A closed box is a cuboid with a base x cm by $2x$ cm and a volume of 576 cm^3 . Its total surface area is $S = 4x^2 + \frac{1728}{x} \text{ cm}^2$. Find the least surface area, and show that it is a minimum. [5 marks]

Answer least surface area = ... cm^2

- D4** The number of fish, P , in a lake t weeks after it was stocked is modelled by $P = 500 + 300\sqrt{t} - 15t$, $t \geq 0$.

- (a) Find $\frac{dP}{dt}$, and the rate at which the number of fish is changing when $t = 25$. Say what this rate means. [4 marks]

Answer fish per week

- (b) Find the greatest number of fish that the model predicts, and show that it is a maximum. [4 marks]

Answer greatest number = ...

- (c) Find when the model predicts $P = 500$ again. Give one reason [2 marks]
why the model may not be suitable after that.

Total for D4: 10 marks

E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

- E1** A cone has its vertex and the whole circle of its base on a sphere [6 marks]
of radius R . Find the greatest possible volume of the cone as a
fraction of the volume of the sphere.

Answer fraction = ...

Extension hints

Cover these with paper. Uncover one at a time.

- E1** **Hint 1** Let the cone have height h and base radius r . Use the centre of the sphere to find r^2 .
- Hint 2** The centre is $h - R$ above the base, so
 $r^2 = R^2 - (h - R)^2 = 2Rh - h^2$, and the
volume is $\frac{\pi}{3}(2Rh^2 - h^3)$.
- Hint 3** $\frac{dV}{dh} = \frac{\pi}{3}(4Rh - 3h^2) = 0$ gives $h = \frac{4R}{3}$.
Then $V = \frac{32\pi R^3}{81}$; divide by $\frac{4}{3}\pi R^3$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can find a rate of change, say what it means, and find when it is greatest. ☐ ☐ ☒

I can use a fixed area or volume to write a quantity in one variable, and make it greatest or least. ☐ ☐ ☒

I can show a value is a maximum or a minimum, and check it makes sense. ☐ ☐ ☒