The Newton–Raphson method

112 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-newton-raphson/

Date

A Do Now

Quick questions to start.

A1 Needed today [3 marks]

Find where the tangent to $y = x^3 - 5$ at the point where $x = 2$ crosses the x-axis.

Answer .....

A2 Needed today [2 marks]

Given $\mathrm{f}(x) = xe^x$, find $\mathrm{f}'(x)$ and the exact value of $\mathrm{f}'(1)$.

Answer .....

A3 Common slip [2 marks]

Given $\mathrm{f}(x) = 3\cos x - x$, find $\mathrm{f}'(1)$ to 4 decimal places. Use radians.

Answer .....

A4 Needed today [3 marks]

Show that $x^3 - 2x - 5 = 0$ has a root between 2 and 2.1.

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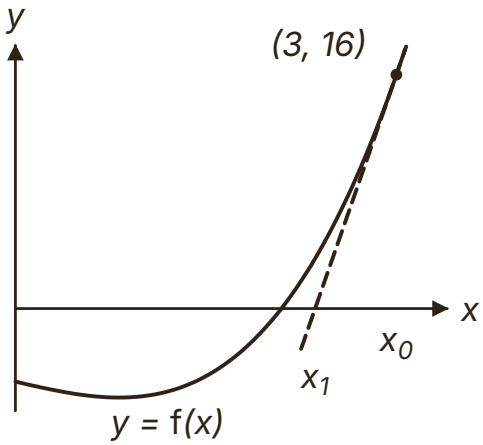
B Example, then your turn

A-Level

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

The diagram shows $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x^3 - 2x - 5$, and the tangent at $x_0 = 3$. The tangent meets the x-axis at x_1. Show that $x_1 = x_0 - \dfrac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)} = 2.36$. Then use $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ to find x_2, x_3 and x_4 to 4 decimal places.



Hint The tangent has gradient $\mathrm{f}'(3) = 25$ and passes through $(3, 16)$. It drops 16 over a horizontal distance of $\dfrac{16}{25}$. Then put the formula into the calculator with Ans and press = again and again.

B1 $\mathrm{f}(x) = x^3 + 3x - 7$. Use the Newton–Raphson method with $x_0 = 1.5$ to find x_1 and x_2 to 4 decimal places. [4 marks]

Answer .....

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EXAMPLE 2

$\mathrm{f}(x) = x - 2\sin x$. Use the Newton–Raphson method with $x_0 = 2$ to find a root of $\mathrm{f}(x) = 0$ to 4 decimal places.

Hint Set the calculator to radians before you start. $\mathrm{f}'(x) = 1 - 2\cos x$. Keep going until two iterates agree to 4 decimal places.

B2 $\mathrm{f}(x) = 3\cos x - x$. Use the Newton–Raphson method with $x_0 = 1$ to find x_1 and x_2 to 4 decimal places. Use radians. [4 marks]

Answer .....

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EXAMPLE 3

$\mathrm{f}(x) = xe^x - 3$. Use the Newton–Raphson method with $x_0 = 1$ to find x_1 and x_2 to 4 decimal places.

Hint Differentiate xe^x with the product rule first: $\mathrm{f}'(x) = (x+1)e^x$.

B3 $\mathrm{f}(x) = x^2 \ln x - 2$, $x > 0$. Find $\mathrm{f}'(x)$. Use the Newton–Raphson method with $x_0 = 1.5$ to find x_1, x_2 and x_3 to 4 decimal places. [5 marks]

Answer .....

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EXAMPLE 4

The equation $e^x - 4x = 0$ has a root α near 2. Use the Newton–Raphson method with $x_0 = 2$ to find x_1, x_2, x_3. Show that $\alpha = 2.153$ to 3 decimal places.

Hint The iterates suggest the value. The proof is a change of sign at the bounds 2.1525 and 2.1535.

B4 The equation $\ln x + x^2 - 5 = 0$ has a root β. Use the Newton–Raphson method with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places. Show that $\beta = 2.067$ to 3 decimal places. [5 marks]

Answer .....

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EXAMPLE 5

$\mathrm{f}(x) = x^2 - 4x + 1$. Explain why $x_0 = 2$ cannot be used to find a root with the Newton–Raphson method. Use $x_0 = 3$ to find x_1, x_2 and x_3.

Hint Work out $\mathrm{f}'(2)$. What does the tangent at a stationary point look like?

B5 $\mathrm{f}(x) = x^3 - 12x + 5$. [4 marks]

Explain why the Newton–Raphson method fails with $x_0 = 2$. Use $x_0 = 3$ to find x_1, x_2 and x_3 to 4 decimal places.

Answer .....

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C1 Sam uses the Newton–Raphson method on $\mathrm{f}(x) = \sin x + x - 2$ with $x_0 = 1$. His calculator is in degrees, and he gets $x_1 = 1.4913$ and $x_2 = 1.7327$. Explain what has gone wrong, then find the correct x_1 and x_2 to 4 decimal places. [5 marks]

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C2 Jo wants the root of $x^3 + x - 5 = 0$ near 1.5. She uses $x_{n+1} = x_n + \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ with $x_0 = 1.5$ and finds $x_1 = 1.4839$, $x_2 = 1.4512$, $x_3 = 1.3838$. Explain her mistake and find the correct x_1 and x_2 to 4 decimal places. [4 marks]

Answer .....

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C3 Kai uses the Newton–Raphson method on $\mathrm{f}(x) = xe^{2x} - 4$ with $x_0 = 0.7$. He writes $\mathrm{f}'(x) = 2e^{2x}$ and gets $x_1 = 0.8432$. Find his mistake. Find the correct $\mathrm{f}'(x)$ and the correct x_1, x_2 and x_3 to 4 decimal places. [5 marks]

Answer .....

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C4 The equation $x^3 - 3x - 1 = 0$ has a root near 1.88. Lena uses the Newton–Raphson method with $x_0 = 0.9$ to find it. Find x_1 to 4 decimal places. Explain, with reference to $\mathrm{f}'(x)$, why her first step goes so far, and state which root her iterates then approach. [5 marks]

D Exam-style questions

Edexcel exam style

D1 $\mathrm{f}(x) = e^{0.5x} - x^2 + 2$. The equation $\mathrm{f}(x) = 0$ has a root α between 2 and 3.

(a) Show that α lies between 2.2 and 2.3. [2 marks]

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(b) Find $\mathrm{f}'(x)$. [2 marks]

Answer .....

(c) Using $x_0 = 2.2$, apply the Newton–Raphson method twice to find x_1 and x_2 to 4 decimal places. [3 marks]

Answer .....

(d) Show that $\alpha = 2.256$ to 3 decimal places. [3 marks]

Total for D1: 10 marks

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D2

> Ola writes: "The equation $x^4 - 4x - 1 = 0$ has a root between 1 and 2. I will start the Newton–Raphson method at $x_0 = 1$, the lower end."

Ola's plan does not work.

(a) Explain why the Newton–Raphson method cannot start at $x_0 = 1$. [2 marks]

(b) Use $x_0 = 1.5$ to find x_1, x_2 and x_3 to 4 decimal places. [3 marks]

Answer .....

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(c) Show that the root is 1.663 to 3 decimal places. [2 marks]

Total for D2: 7 marks

D3 The equation $e^{-x} + x^2 = 3$ has a positive root α. Starting with $x_0 = 1.5$, use the Newton–Raphson method to find α correct to 3 decimal places. Show that your answer is correct to this accuracy. [7 marks]

Answer .....

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D4 A drone flies above a lake. Its height, h metres, t seconds after it leaves a jetty is modelled by $h = 25\sin\left(\frac{t}{8}\right) + 2 - 0.8t$, where the angle is in radians. The drone reaches the water at time T seconds.

(a) Show that T lies between 20 and 21. [2 marks]

(b) Find $\frac{dh}{dt}$. [2 marks]

Answer .....

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(c) Use the Newton–Raphson method with $t_0 = 20$ to find T to 2 decimal places. Show that your answer is correct. [3 marks]

Answer .....

(d) Use the model to find the greatest height of the drone, to 3 significant figures. [4 marks]

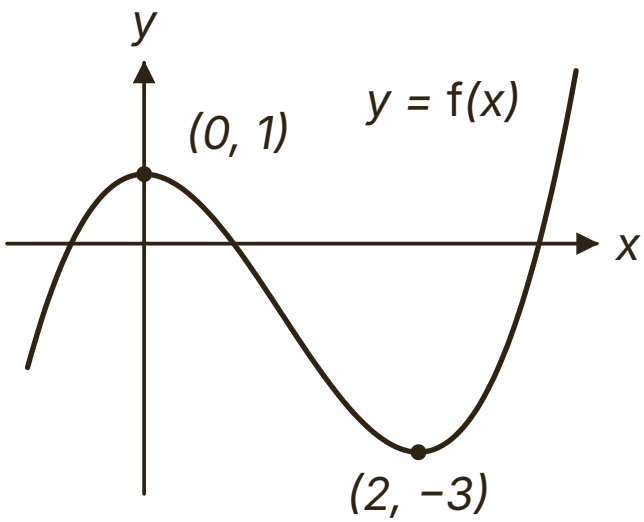
Answer .....

Total for D4: 11 marks

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D5 The diagram shows $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x^3 - 3x^2 + 1$. The curve has turning points at $(0, 1)$ and $(2, -3)$, and crosses the x-axis three times.

(a) Explain why $x_0 = 2$ cannot be used to find the largest root with the Newton–Raphson method. [2 marks]



(b) Find x_1 when $x_0 = 2.1$, to 4 decimal places. Explain why x_1 is so far from x_0. [3 marks]

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(c) Use $x_0 = 3$ to find the largest root to 4 decimal places. [2 marks]

Answer .....

(d) Starting from $x_0 = 1$, find x_1 and x_2 and state which root the iterates approach. [2 marks]

Answer .....

Total for D5: 9 marks

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D6 $\mathrm{f}(x) = x^3 - 4x - 2$. The equation $\mathrm{f}(x) = 0$ has a root α near 2.

(a) Show that the Newton–Raphson formula for this equation can be written $x_{n+1} = \dfrac{2x_n^3 + 2}{3x_n^2 - 4}$. [3 marks]

(b) Use this formula with $x_0 = 2$ to find x_1, x_2 and x_3 to 4 decimal places. [3 marks]

Answer .....

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(c) Show that $\alpha = 2.214$ to 3 decimal places. [2 marks]

Total for D6: 8 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 [9 marks]

The functions $\mathrm{f}(x) = \sqrt[3]{x - 2}$ and $\mathrm{g}(x) = (x - 2)^3$ both have the root $x = 2$. Apply the Newton–Raphson method to each.

For f, start at $x_0 = 2.1$, find x_1, x_2, x_3, and show that the iterates move away from 2 for every starting value other than 2.

For g, start at $x_0 = 3$ and find the smallest n for which x_n is within 0.001 of 2.

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Write $e_n = x_n - 2$, the distance from the root. Simplify $\dfrac{\mathrm{f}(x)}{\mathrm{f}'(x)}$ first.

Hint 2 For f, $\dfrac{\mathrm{f}(x)}{\mathrm{f}'(x)} = 3(x-2)$. For g, $\dfrac{\mathrm{g}(x)}{\mathrm{g}'(x)} = \dfrac{x-2}{3}$.

Hint 3 Find how e_{n+1} is related to e_n in each case. For g you need $\left(\dfrac{2}{3}\right)^n < 0.001$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can use the Newton–Raphson formula to find a root, using the full calculator value each time.	○ ○ ○
I can differentiate the function correctly first, with radians for trigonometric functions.	○ ○ ○
I can show that a root is correct to a stated accuracy with a change of sign.	○ ○ ○
I can explain why the method fails at or near a stationary point.	○ ○ ○

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