Applications to modelling

122 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-numerical-modelling/

Date

A Do Now

Quick questions to start.

A1 Needed today [3 marks]

Given $\mathrm{f}(x) = x^2 e^{-0.5x}$, find $\mathrm{f}'(x)$ and the value of $\mathrm{f}'(2)$ to 4 decimal places.

Answer .....

A2 Last lesson [3 marks]

Use the Newton–Raphson method on $\mathrm{f}(x) = x^3 - 10$ with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

Answer .....

A3 Last lesson [2 marks]

Use $x_{n+1} = 3 + \ln x_n$ with $x_0 = 4$ to find x_1, x_2 and x_3 to 4 decimal places.

Answer .....

A4 Common slip [2 marks]

A model gives a time of $t = 4.37$ hours. Write this time in hours and minutes, to the nearest minute.

Answer .....

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B Example, then your turn

A-Level

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

The value of a van, $£V$, t years after it is bought, is modelled by $V = 18\,000e^{-0.2t} - 400t$. (a) Find the value after 3 years, to the nearest $£100$. (b) Show that the model gives a value of zero between $t = 8$ and $t = 9$. (c) Use the Newton–Raphson method with $t_0 = 8.5$ to find t_1 to 3 decimal places. (d) Give one criticism of the model for large t.

Hint Write $\mathrm{f}(t) = 18\,000e^{-0.2t} - 400t$. Then $\mathrm{f}'(t) = -3600e^{-0.2t} - 400$. A value cannot be negative.

B1 [6 marks]

The value of a bike, $£B$, t years after it is bought, is modelled by $B = 3000(0.78)^t - 150t$. Show that $B = 0$ between $t = 5$ and $t = 6$. Use the Newton–Raphson method with $t_0 = 5.5$ to find t_1 to 3 decimal places. Explain why the model is not valid for $t > t_1$.

Answer .....

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EXAMPLE 2

A runner's speed, v m/s, t seconds after the start is modelled by $v = (8 - 0.5t)\ln(t+1)$, $0 \le t \le 16$. (a) Show that $\dfrac{dv}{dt} = 0$ can be written $t = \dfrac{17}{1+\ln(t+1)} - 1$. (b) Use $t_{n+1} = \dfrac{17}{1+\ln(t_n+1)} - 1$ with $t_0 = 5$ to find the time of the greatest speed to 3 decimal places. (c) Find the greatest speed.

Hint Product rule: $\dfrac{dv}{dt} = -0.5\ln(t+1) + \dfrac{8-0.5t}{t+1}$. Multiply by $2(t+1)$, then add $t+1$ to both sides.

B2 [7 marks]

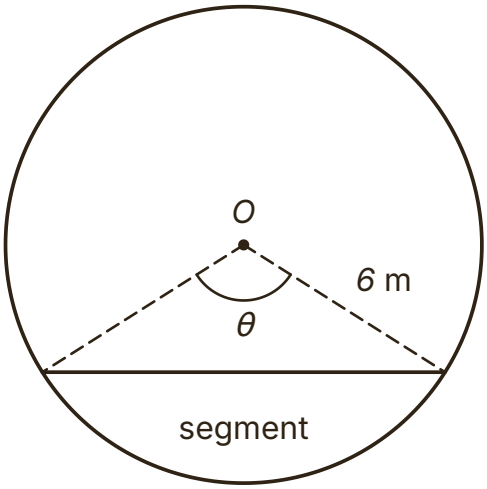
The power output, P kW, of a machine t hours after it starts is modelled by $P = (12 - t)\ln(t+2)$, $0 \le t \le 12$. Show that $\dfrac{dP}{dt} = 0$ can be written $t = \dfrac{14}{1+\ln(t+2)} - 2$. Use this iteration with $t_0 = 3$ to find the time of the greatest output to 3 decimal places, and find the greatest output.

Answer .....

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EXAMPLE 3

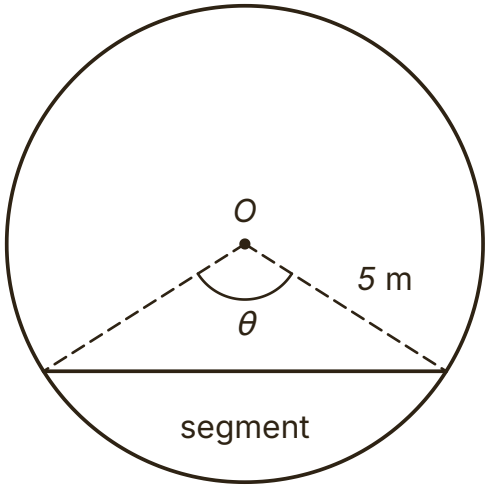
A circular pond has radius 6 m. A straight bridge cuts off a segment of the pond with area 20 m². The angle at the centre is θ radians. (a) Show that $\theta - \sin\theta = \dfrac{10}{9}$. (b) Use the Newton–Raphson method with $\theta_0 = 2$ to find θ to 4 decimal places. (c) Find the length of the bridge.



Hint The area of a segment is $\dfrac{1}{2}r^2(\theta - \sin\theta)$. The chord has length $2r\sin\left(\dfrac{\theta}{2}\right)$.

B3 [6 marks]

A circular lawn has radius 5 m. A straight path cuts off a segment of area 15 m², with angle θ radians at the centre. Show that $\theta - \sin\theta = 1.2$. Use the Newton–Raphson method with $\theta_0 = 2$ to find θ to 3 decimal places, and find the length of the path.



not to scale

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Answer .....

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EXAMPLE 4

The concentration, C mg/l, of a painkiller in the blood t hours after a dose is $C = 10te^{-0.4t}$. It works while $C \geq 3$. The concentration falls back to 3 at time T, where T is near 8. Use the Newton–Raphson method with $t_0 = 8$ to find T, and show that the painkiller stops working at 8 hours 18 minutes, to the nearest minute.

Hint Use $\mathrm{f}(t) = 10te^{-0.4t} - 3$ and $\mathrm{f}'(t) = (10 - 4t)e^{-0.4t}$. 8 h 18 min to the nearest minute means a time between 8 h 17.5 min and 8 h 18.5 min.

B4 [6 marks]

The amount of a pollutant in a river, P units, t days after a spill is $P = 25te^{-0.3t}$. The river is safe when $P < 5$. The amount falls to 5 at time T days, where T is near 14. Use the Newton–Raphson method with $t_0 = 14$ to find T to 4 decimal places. Show that the river becomes safe 14 days 5 hours after the spill, to the nearest hour.

Answer .....

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C1 The depth of water in a harbour, d metres, t hours after midnight is modelled by $d = 5 + 2\sin(0.5t) - 0.1t$, with the angle in radians. A boat needs a depth of at least 4 m. Mia solves $\mathrm{f}(t) = 1 + 2\sin(0.5t) - 0.1t = 0$ with the Newton–Raphson method and $t_0 = 7$. Her calculator is in degrees and she gets $t_1 = 6.5300$ and $t_2 = 6.0170$. Explain her mistake. Show that the correct root lies between 6 and 7, find the correct t_1 and t_2 to 4 decimal places, and give the time the depth first falls to 4 m, to the nearest minute. [7 marks]

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C2 A tank starts filling at $09{:}00$. The volume of water, V litres, t hours later is $V = 120(1 - e^{-0.3t}) + 8t$. An alarm sounds when $V = 150$. Ali finds the root $t = 6.1327$ and says the alarm sounds at $15{:}13$. Explain Ali's mistake. Show that the alarm sounds at $15{:}08$, to the nearest minute. [5 marks]

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C3 A section of a roller coaster track has height $h = 30 + 8\sin(0.4x) - 0.05x^2$ metres, x metres along the ground from the start, $0 \le x \le 16$. The track has a lowest point at $x = \beta$, where β is near 10. Find $\dfrac{dh}{dx}$ and $\dfrac{d^2h}{dx^2}$. Apply the Newton–Raphson method to $\dfrac{dh}{dx} = 0$ with $x_0 = 10$ to find β to 3 decimal places. Find the height at this point and show that it is a minimum. [7 marks]

Answer .....

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C4 A ball is thrown upwards from a cliff top. Its height above the sea, h metres, t seconds later is modelled by $h = 2 + 14t - 4.9t^2 - 0.5e^{0.3t}$. Leo uses the Newton–Raphson method with $t_0 = 0$ and gets $t_2 = -0.1044$. He says the ball lands at -0.10 s. Explain why his answer is not valid. Show that the ball lands between $t = 2.5$ and $t = 3$, then use $t_0 = 3$ to find the time it lands, to 2 decimal places. [6 marks]

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D1 The concentration, C mg/l, of a drug in a patient's blood t hours after an injection is modelled by $C = 6(e^{-0.2t} - e^{-0.7t})$. The drug is effective while $C \geq 2$.

(a) Find $\dfrac{dC}{dt}$ and show that the concentration is greatest when $t = 2\ln 3.5$. [3 marks]

(b) The concentration falls back to 2 at time T. Show that T lies between 5 and 5.5. [2 marks]

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(c) Use the Newton–Raphson method with $t_0 = 5.5$ to find t_1 and t_2 to 4 decimal places. [3 marks]

Answer .....

(d) The concentration first reaches 2 at $t = 1.0631$. Find, to the nearest minute, how long the drug is effective. [2 marks]

Answer .....

Total for D1: 10 marks

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D2

> Jo models the temperature in a greenhouse, $T\,^\circ$C, t hours after midnight by $T = 3 + 0.12(t - 6)^2 + \ln(t + 1)$, $0 \leq t \leq 12$. She writes: "$T(0) = 7.32$ and $T(12) = 9.88$. Both are above 5 and there is no change of sign, so the temperature never falls below $5\,^\circ$C."

Jo is wrong.

(a) Explain why Jo's argument is not valid. [2 marks]

(b) By evaluating T at $t = 4$, 5 and 7, show that the temperature falls below $5\,^\circ$C and then rises above it. [3 marks]

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(c) The two times are 4.3683 and 6.3063 hours after midnight. Find, to the nearest minute, how long the temperature is below $5\ ^\circ\text{C}$. [3 marks]

Answer .....

Total for D2: 8 marks

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D3 Kepler's equation links the position of a planet in its orbit to time: $M = E - 0.25\sin E$, where the angles are in radians. When $M = 1.2$, find E correct to 3 decimal places. Use the Newton–Raphson method with $E_0 = 1.2$, and show that your answer is correct to this accuracy. [7 marks]

Answer .....

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D4 A water rocket is launched from the ground. Its height, h metres, t seconds after launch is modelled by $h = 30t - 5t^2 - 2(e^{0.4t} - 1)$, $t \geq 0$. The rocket lands at time T seconds.

(a) Show that T lies between 5 and 5.5. [2 marks]

(b) Find $\dfrac{dh}{dt}$. [2 marks]

Answer .....

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(c) Use the Newton–Raphson method with $t_0 = 5$ to find T to 2 decimal places. Show that your answer is correct to this accuracy. [3 marks]

Answer .....

(d) The greatest height is reached at $t = 2.7588$. Find the greatest height to 3 significant figures, and give one reason why the model may not be suitable. [2 marks]

Total for D4: 9 marks

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D5 A species of fish, A, lives in a lake. At $t = 0$ a second species, B, is introduced. The numbers of each, t years later, are modelled by $A = 50e^{0.12t}$ and $B = 300\ln(t + 1)$.

(a) Show that $A = B$ for a value of t between 24 and 25. [2 marks]

(b) Use the Newton–Raphson method with $t_0 = 25$ to find this value of t to 2 decimal places. [3 marks]

Answer .....

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(c) The equation $A = B$ has a second root at $t = 0.186$. Describe what happens to the two populations between $t = 0$ and $t = 30$. [2 marks]

(d) Give one reason why model A is not realistic for large t. [2 marks]

Total for D5: 9 marks

D6 A company makes x hundred items a week. Its weekly profit, $£P$ thousand, is modelled by $P = 40xe^{-0.1x} - 2x - 30$.

(a) Show that the company stops making a profit at a value of x between 25 and 26. [2 marks]

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(b) Show that $\frac{dP}{dx} = 0$ can be written $x = 10 - 0.5e^{0.1x}$. [3 marks]

(c) Use $x_{n+1} = 10 - 0.5e^{0.1x_n}$ with $x_0 = 7$ to find x_1, x_2, x_3 and x_4 to 4 decimal places. [3 marks]

Answer .....

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(d) The iterates converge to 8.795. Find the greatest weekly profit, to the nearest $£100$. [2 marks]

Answer .....

Total for D6: 10 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 A horizontal pipe has a circular cross-section of radius 40 cm. Water flows along the bottom of the pipe and fills 30% of the area of the cross-section. Find the depth of the water, to the nearest millimetre. Show that your answer is correct to this accuracy. [9 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 The water is a segment of the circle. Let θ be the angle it makes at the centre. Its area is $\frac{1}{2}r^2(\theta - \sin\theta)$.

Hint 2 $\frac{1}{2}r^2(\theta - \sin\theta) = 0.3\pi r^2$ gives $\theta - \sin\theta = 0.6\pi$. Use the Newton–Raphson method, starting near 2.5.

Hint 3 The depth is $40 - 40\cos\left(\frac{\theta}{2}\right)$. To check the depth to the nearest millimetre, work out the fraction of the area filled at depths of 27.15 cm and 27.25 cm.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can set up an equation f(x) = 0 from a model and find a root numerically.	○ ○ ○
I can give a root in the units of the problem, such as hours and minutes.	○ ○ ○
I can show that a time or length is correct to the accuracy asked for.	○ ○ ○
I can check that a root makes sense in context and criticise the model.	○ ○ ○

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