Proof by contradiction

112 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/a-level-proof-by-contradiction/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Prove that the square of any odd integer is odd.

A2 Common slip [2 marks]

You want to prove "if n^2 is even, then n is even" by contradiction. Write down the assumption you start from.

A3 Needed today [2 marks]

Complete the definition: a number x is rational if ...

A4 Needed today [3 marks]

Prove that $n^3 - n$ is even for every integer n.

A5 Common slip [2 marks]

Amir says $n^2 + n + 41$ is prime for every positive integer n. Show that he is wrong.

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B Example, then your turn

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Prove by contradiction that $\sqrt{2}$ is irrational.

Hint Assume $\sqrt{2} = \frac{a}{b}$ with a and b integers with no common factor. Show that a and b are both even.

B1 Prove by contradiction that $\sqrt{3}$ is irrational. You may use the fact that if a^2 is a multiple of 3 then a is a multiple of 3. [5 marks]

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EXAMPLE 2

Prove that if n^2 is a multiple of 3, then n is a multiple of 3.

Hint Assume n is not a multiple of 3. Then $n = 3k + 1$ or $n = 3k + 2$. Square each one.

B2 Prove by contradiction that if n^3 is even, then n is even. [4 marks]

EXAMPLE 3

Prove by contradiction that there are infinitely many prime numbers.

Hint Assume there is a largest list $p_1, p_2, \ldots, p_n$ of all the primes. Look at $N = p_1 p_2 \cdots p_n + 1$.

B3 Prove by contradiction that there is no smallest positive rational number. [4 marks]

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EXAMPLE 4

Prove by contradiction that there are no positive integers x and y with $x^2 - y^2 = 1$.

Hint Factorise: $(x - y)(x + y) = 1$. Both brackets are integers and $x + y \geq 2$.

B4 Prove by contradiction that there are no positive integers x and y with $x^2 - y^2 = 10$. [5 marks]

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EXAMPLE 5

Given that $\sqrt{2}$ is irrational, prove that $3 + \sqrt{2}$ is irrational.

Hint Assume $3 + \sqrt{2} = q$, a rational number. Rearrange to make $\sqrt{2}$ the subject.

B5 Given that $\sqrt{5}$ is irrational, prove by contradiction that $\dfrac{2}{\sqrt{5} - 1}$ is irrational. [5 marks]

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C1 A student writes: "Assume $\sqrt{6} = \dfrac{a}{b}$ with a and b integers. Then $a^2 = 6b^2$, so a is even, $a = 2c$. Then $2c^2 = 3b^2$, so b is even. That is a contradiction, so $\sqrt{6}$ is irrational." Explain why the argument does not prove the result, and then correct it. [4 marks]

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C2 Ella wants to prove "for every integer n, $n^2 - 2$ is not a multiple of 4". She starts: "Assume that for every integer n, $n^2 - 2$ is a multiple of 4." Explain what is wrong with her assumption. Then give a full proof by contradiction. [5 marks]

C3 Prove by contradiction that $\sqrt[3]{2}$ is irrational. [5 marks]

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C4 Prove by contradiction that $\log_2 3$ is irrational. [5 marks]

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D Exam-style questions

Edexcel exam style

D1 Prove by contradiction that there are no integers p and q such that $9p^2 - q^2 = 16$. [9 marks]

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D2 In this question n is an integer.

(a) Prove that if n^2 is a multiple of 5 then n is a multiple of 5. [4 marks]

(b) Hence prove by contradiction that $\sqrt{5}$ is irrational. [5 marks]

Total for D2: 9 marks

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D3

> Kai says: "The proof tells us that N is always a prime number."

Kai is shown a proof that there are infinitely many primes. It uses $N = p_1 p_2 \cdots p_n + 1$, where $p_1, \ldots, p_n$ are the first n primes.

(a) Explain why this does not mean N is always prime, and say what the proof does show about N. [3 marks]

(b) Show that $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$ is not prime. [3 marks]

Answer .....

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(c) Explain why neither 59 nor 509 is in the list $2, 3, 5, 7, 11, 13,$ using the way N was built. [3 marks]

Total for D3: 9 marks

D4 In this question x, a and b are positive real numbers.

(a) Prove by contradiction that $x + \dfrac{1}{x} \geq 2$. [4 marks]

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(b) Hence show that $\frac{a}{b} + \frac{b}{a} \geq 2$. [2 marks]

(c) Prove that $\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) \geq 4$. [3 marks]

Total for D4: 9 marks

D5 In this question n, x and y are integers.

(a) Prove that $n^2 - 2$ is never a multiple of 3. [4 marks]

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(b) Hence prove by contradiction that there are no integers x and y with $x^2 + 3y = 3x + 5$. [4 marks]

Total for D5: 8 marks

D6 Given that $\sqrt{6}$ is irrational, prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational. [6 marks]

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 Prove that if a, b and c are odd integers, then the equation $ax^2 + bx + c = 0$ has no rational solutions. [9 marks]

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Suppose $x = \frac{p}{q}$ is a root, with p and q integers that have no common factor. Multiply the equation by q^2.

Hint 2 You now have $ap^2 + bpq + cq^2 = 0$. Since p and q have no common factor, they are not both even. That leaves three cases.

Hint 3 In each case work out whether each of the three terms is odd or even, using that a, b, c are odd. What is their sum?

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can write the correct assumption to start a proof by contradiction.	○ ○ ○
I can prove that roots and logarithms are irrational.	○ ○ ○
I can prove that equations have no integer solutions.	○ ○ ○
I can find and correct the errors in a proof.	○ ○ ○

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