



Algebraic proof

About 80 minutes No calculator 70 marks

Date _____

A Do Now

The first two questions are what today's lesson needs. The others bring back earlier topics.

- A1** Needed today [2 marks]

Factorise fully $5x^2 - 45$.

A2 Needed today [2 marks]

Write $x^2 + 8x + 27$ in the form $(x + a)^2 + b$.

Answer

Answer $(x + \dots)^2 + \dots$

- A3** Last lesson [2 marks]

Solve $8^x = 32$.

A4 From chapter 4 [2 marks]

Solve $x^2 - x - 12 < 0$.

Answer $x = \dots$

Answer

B Example, then your turn

Level 2

Your teacher works through each example with you. Then try the one beside it.

EXAMPLE 1

Prove that $4(2n + 9) - 2(n - 3)$ is a multiple of 6 for every integer n .

Hint Expand both brackets and collect like terms. Then take out the common factor.

- B1** Prove that $6(2n + 5) - 3(n - 2)$ [3 marks] is a multiple of 9 for every integer n .

EXAMPLE 2

Prove that $4n^2(16n - 3) + 12n^2$ is a cube number when n is a positive integer.

Hint Expand the bracket and simplify. Then write what is left as something cubed.

- B2** Prove that $3k(12k + 5) - 15k$ [3 marks] is a square number when k is a positive integer.

EXAMPLE 3

Write $x^2 - 12x + 41$ in the form $(x - a)^2 + b$.
Hence prove that $x^2 - 12x + 41 > 0$ for all values of x .

Hint Halve the 12 to get the number in the bracket, then take off 6^2 . A square is never negative.

B3 Prove that $x^2 + 14x + 51 > 0$ [4 marks]
for all values of x .

EXAMPLE 4

a and b are positive integers with $a > b$. Prove that $\frac{ax^2 + 5ax}{bx^2 + 5bx} > 1$, where $x \neq 0$ and $x \neq -5$.

Hint Factorise the top and the bottom. Look for the factors they share.

B4 Prove that $\frac{12 - 4y}{y^2 - 9}$ is negative [4 marks]
for every $y > -3$, where $y \neq 3$.

EXAMPLE 5

$f(n) = n^2 + 3n$. Prove that $f(n + 1) - f(n - 1)$ is always even.

Hint Put $n + 1$, then $n - 1$, in place of n . Expand, subtract, and take out a factor of 2.

B5 $f(x) = x^2 + 4x$. Prove that [3 marks]
 $f(3x) - f(x) = kx(x + 1)$,
where k is an integer.

c Practice

Level 2

C1 Prove that $3(4n - 1) - 4(3n - 5)$ [2 marks]
is the same positive integer for
every value of n .

C2 Prove that the product of two [3 marks]
consecutive odd numbers is
one less than a multiple of 4.

- C3** $f(x) = (2x - 5)^2 + 8(2x - 3)$. [3 marks] **C4** Prove that $10x - x^2 - 30$ is [3 marks]
Prove that there is exactly one
value of x for which $f(x) = 0$.
- C5** Prove that $\frac{15p - 20}{3p - 4}$ is always a [2 marks] **C6** p is a positive number and q is a [3 marks]
positive integer, where $p \neq \frac{4}{3}$.
negative number. Prove that
 $\frac{3p^2 - 3pq}{pq - q^2}$ is negative.

D Exam-style questions

AQA exam style

D1 Priya says, " $(n + 3)^2 - n^2$ is a multiple of 6 for every positive integer n ."

n is a positive integer.

(a) Prove that $(n + 3)^2 - n^2$ is a multiple of 3. [3 marks]

(b) Explain why Priya is wrong. [1 mark]

Total for D1: 4 marks

D2 The n th term of a sequence is $n^2 + 2n$.

(a) Work out an expression for the $(n + 1)$ th term of the sequence. Give your [2 marks]
answer in its simplest form.

Answer

(b) Hence prove that the difference between any two consecutive terms of the [2 marks]
sequence is an odd number.

- (c) Prove that one more than any term of the sequence is a square number. [2 marks]

Total for D2: 6 marks

D3 $f(x) = 2x^2 - 16x + 37$

- (a) Write $f(x)$ in the form $a(x + b)^2 + c$, where a , b and c are integers. [3 marks]

Answer $(x + \dots\dots)^2 + \dots\dots$

- (b) Hence prove that $f(x) > 0$ for all values of x . [1 mark]

- (c) Hence work out the greatest value of $\frac{10}{f(x)}$. [2 marks]

Answer

Total for D3: 6 marks

- D4** Prove that $\frac{2x^2 + 6x}{x^2 - 9} \div \frac{x^2 + x}{x^2 - 2x - 3} = 2$ for every value of x for which it is defined. [4 marks]
You must show your working.

D5 $g(n) = n^2 - 2n$

- (a) Show that $g(n + 1) = n^2 - 1$. [2 marks]

- (b) Hence prove that $g(n + 1)$ is a multiple of 8 when n is an odd positive integer. [3 marks]

Total for D5: 5 marks

No route is given. The hints are at the end of the sheet. Use one at a time.

- E1** Prove that the difference between the cubes of two consecutive odd numbers is [4 marks]
always 2 more than a multiple of 24.

Extension hints

Cover these with paper. Uncover one at a time.

- E1** **Hint 1** Write the two odd numbers as $2n - 1$ and $2n + 1$, and expand each cube.
- Hint 2** $(2n + 1)^3 = 8n^3 + 12n^2 + 6n + 1$ and
 $(2n - 1)^3 = 8n^3 - 12n^2 + 6n - 1$.
- Hint 3** Subtract the second cube from the first: the $8n^3$ and $6n$ terms cancel, leaving $24n^2 + 2$. That is 2 more than $24n^2$, a multiple of 24.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can prove a result about integers by writing an expression as a multiple, a square or a cube. ☐ ☐ ☐

I can complete the square to prove an expression is always positive or always negative. ☐ ☐ ☐

I can factorise and cancel algebraic fractions to prove a result. ☐ ☐ ☐

I can use function notation and forms such as $2n + 1$ in a proof. ☐ ☐ ☐