



Differentiation, tangents and normals, and stationary points

About 90 minutes No calculator 113 marks

Date _____

A Do Now

The first one is what today needs. The others bring back earlier work.

A1 Needed today [2 marks]

Write $\frac{3x^5 - 8}{2x^2}$ as a sum of terms of the form ax^n .

Answer

A2 From chapter 5 [2 marks]

Work out the equation of the line through $(4, -3)$ that is perpendicular to $y = \frac{2}{5}x + 1$. Give it in the form $y = mx + c$.

Answer

A3 From chapter 4 [2 marks]

Solve $3x^2 - 13x - 10 \leq 0$.

Answer

A4 From chapter 2 [2 marks]

Write $x^2 - 14x + 58$ in the form $(x - a)^2 + b$.

Answer

Solve $2x = \frac{54}{x^2}$.

Answer

B Examples, then your turn

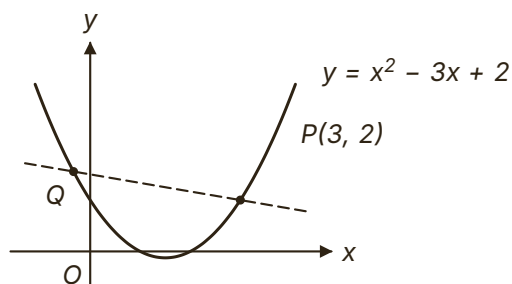
Level 2

Your teacher works through each example with you. Then try the one beside it on your own.

EXAMPLE 1

The normal to the curve $y = x^2 - 3x + 2$ at the point $P(3, 2)$ meets the curve again at Q .

Work out the coordinates of Q .



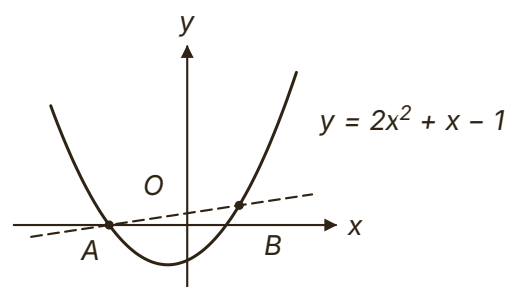
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Hint $\frac{dy}{dx} = 2x - 3 = 3$ at P , so the normal has gradient $-\frac{1}{3}$: $y - 2 = -\frac{1}{3}(x - 3)$, which is $y = -\frac{1}{3}x + 3$. Where it meets the curve, $x^2 - 3x + 2 = -\frac{1}{3}x + 3$, so $3x^2 - 8x - 3 = 0$, which is $(3x + 1)(x - 3) = 0$. $x = 3$ is P , so Q has $x = -\frac{1}{3}$ and $y = \frac{1}{9} + 1 + 2 = \frac{28}{9}$.

B1 The normal to the curve [5 marks]

$y = 2x^2 + x - 1$ at the point A where $x = -1$ meets the curve again at B .

Work out the coordinates of B .



Not drawn accurately

Answer $B(\dots, \dots)$

EXAMPLE 2

The curve $y = x^3 + ax^2 + bx + 4$ has a stationary point at $(3, -5)$.

Work out the values of a and b . Work out the coordinates of the other stationary point and decide whether each point is a maximum or a minimum.

Hint At $(3, -5)$: $27 + 9a + 3b + 4 = -5$, so $3a + b = -12$. $\frac{dy}{dx} = 3x^2 + 2ax + b = 0$ at $x = 3$: $6a + b = -27$. Subtracting, $3a = -15$: $a = -5$, $b = 3$. Then $\frac{dy}{dx} = 3x^2 - 10x + 3 = (3x - 1)(x - 3)$, so the other point has $x = \frac{1}{3}$ and $y = \frac{121}{27}$. $\frac{d^2y}{dx^2} = 6x - 10$ is $8 > 0$ at $x = 3$ (a minimum) and $-8 < 0$ at $x = \frac{1}{3}$ (a maximum).

B2 The curve

[6 marks]

$y = ax^3 + bx^2 - 6x$ has a stationary point at $(1, -5)$.

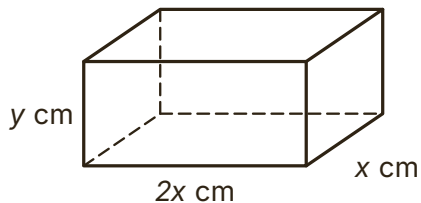
Work out the values of a and b . Work out the coordinates of the other stationary point and decide whether each point is a maximum or a minimum.

Answer $a = \dots\dots$, $b = \dots\dots$; $(\dots\dots, \dots\dots)$

EXAMPLE 3

A closed box is a cuboid. Its base is x cm by $2x$ cm and its height is y cm. Its volume is 72 cm^3 .

Show that its surface area, $A \text{ cm}^2$, is $A = 4x^2 + \frac{216}{x}$. Work out the least surface area and show that it is a minimum.



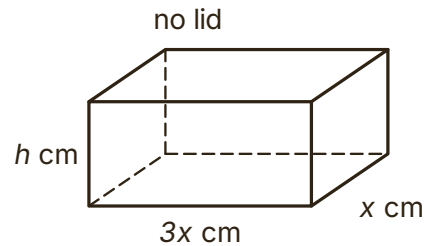
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Hint $2x^2y = 72$, so $y = \frac{36}{x^2}$.

$A = 2(2x^2) + 2(xy) + 2(2xy)$, which is $4x^2 + 6xy = 4x^2 + \frac{216}{x}$. $\frac{dA}{dx} = 8x - \frac{216}{x^2} = 0$ when $x^3 = 27$, so $x = 3$. $\frac{d^2A}{dx^2} = 8 + \frac{432}{x^3} > 0$, a minimum. $A = 36 + 72 = 108$.

B3 An open box has no lid. Its base is x cm by $3x$ cm and its height is h cm. Its volume is 144 cm^3 . [6 marks]

Show that the area of card used, $A \text{ cm}^2$, is $A = 3x^2 + \frac{384}{x}$. Work out the least area of card and show that it is a minimum.



Not drawn accurately

Answer cm^2

C Practice

Level 2

Each question asks for something different. Show your working.

C1 $P(3, 15)$ is a point on the curve $y = x^3 - 4x$. Q is the point on the curve with x -coordinate $3 + h$. [3 marks]

Show that the gradient of the chord PQ is $23 + 9h + h^2$. Hence write down the gradient of the curve at P .

Answer

C2 $y = \frac{(x^2 - 3)^2}{x}$. Work out $\frac{dy}{dx}$.

[3 marks]

Answer $\frac{dy}{dx} = \dots\dots$

C3 Work out the coordinates of the point on the curve $y = 3x + \frac{16}{x^2}$ where the gradient is 7.

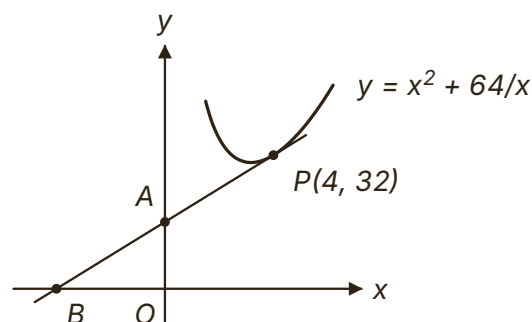
[3 marks]

Answer $(\dots\dots, \dots\dots)$

C4 The tangent to the curve $y = x^2 + \frac{64}{x}$ at the point $P(4, 32)$ meets the y -axis at A and the x -axis at B .

[4 marks]

Work out the area of triangle OAB .



Answer $\dots\dots$

Not drawn accurately

C5 Work out the equation of the normal to the curve $y = x^3 - 4x^2 + 7$ at the point where $x = 3$. Give your answer in the form $ax + by + c = 0$, where a , b and c are integers.

[4 marks]

Answer $\dots\dots$

- C6** Work out the values of x for which $y = 4x^3 - 3x^2 - 18x + 1$ is a decreasing function. [3 marks]

Answer

- C7** $f(x) = x^3 - 6x^2 + 15x - 4$. Use differentiation to show that $f(x)$ is an increasing function for all values of x . [3 marks]

- C8** $y = (2x - 1)^2(x + 3)$. Work out $\frac{d^2y}{dx^2}$. Hence work out the rate of change of the gradient of the curve at the point where $x = -2$, and say whether the gradient is increasing or decreasing there. [3 marks]

Answer

- C9** $y = x^3 + \frac{48}{x}$ for $x > 0$. Work out the coordinates of the stationary point and decide whether it is a maximum or a minimum. [4 marks]

Answer (.....,)

Answer every question in the space given. Show your working: most of the marks are for method.

D1

Leo is finding the stationary points of $y = x^4 - 4x^3$. He writes:

" $\frac{dy}{dx} = 4x^3 - 12x^2 = 0$. Dividing by $4x^2$ gives $x - 3 = 0$, so the curve has only one stationary point, where $x = 3$."

Leo's conclusion is wrong.

(a) Explain Leo's mistake. [1 mark]

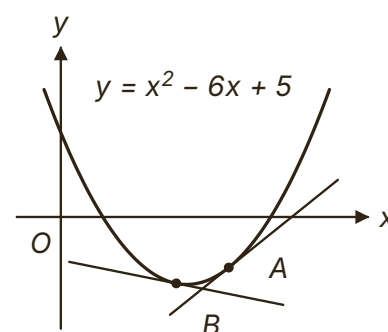
(b) Work out the coordinates of both stationary points. Decide whether each is a maximum, a minimum or neither. You must show your working. [5 marks]

Answer (.....,) and (.....,)

Total for D1: 6 marks

D2 The curve $y = x^2 - 6x + 5$ passes through the point $A(4, -3)$.

(a) Work out the equation of the tangent to the curve at A . Give your answer in the form $y = mx + c$. [3 marks]



Answer $y = \dots\dots$

(b) B is the point on the curve where the tangent is perpendicular to the tangent at A . Work out the coordinates of B . [3 marks]

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Answer $B(\dots\dots, \dots\dots)$

Total for D2: 6 marks

D3 A particle moves along a straight line. Its displacement from a fixed point O after t seconds is s metres, where $s = t^3 - 9t^2 + 24t$ for $t \geq 0$.

(a) Work out the times at which the particle is at rest.

[3 marks]

Answer $t = \dots\dots$

(b) Write down the values of t for which the displacement is decreasing.

[1 mark]

Answer $\dots\dots$

(c) The acceleration of the particle is $\frac{d^2s}{dt^2}$ m/s². Work out the acceleration when $t = 5$.

[2 marks]

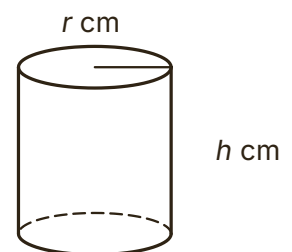
Answer $\dots\dots$ m/s²

Total for D3: 6 marks

D4 A closed cylinder has radius r cm and height h cm. Its total surface area is 54π cm².

(a) Show that the volume of the cylinder, V cm³, is $V = 27\pi r - \pi r^3$.

[2 marks]



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(b) Use calculus to work out the greatest volume of the cylinder. Show that it is a maximum. Give your answer in terms of π .

[4 marks]

Answer $\dots\dots$ cm³

Total for D4: 6 marks

D5 A curve has equation $y = ax + \frac{b}{x^2}$, where a and b are constants. The curve passes through the point $(1, 10)$, and its gradient at that point is -14 .

(a) Work out the values of a and b .

[3 marks]

Answer $a = \dots\dots$, $b = \dots\dots$

(b) Hence work out the coordinates of the stationary point on the curve and decide whether it is a maximum or a minimum. [2 marks]

Answer $(\dots\dots, \dots\dots)$

Total for D5: 5 marks

D6 $y = x^3 + kx^2 + 3x$, where k is a constant. $\frac{dy}{dx} > 0$ for all values of x .

[4 marks]

Work out the range of possible values of k .

Answer $\dots\dots$

D7 The line $y = 9x - 16$ is a tangent to the curve $y = x^3 + kx$ at the point P , where k is a constant. [5 marks]

Work out the value of k and the coordinates of P .

Answer $k = \dots\dots$, $P(\dots\dots, \dots\dots)$

D8 $y = \frac{x^2 + 2}{x}$. Which of these is $\frac{dy}{dx}$? Circle your answer. [1 mark]

$1 - \frac{2}{x^2}$ $1 + \frac{2}{x^2}$ $1 - \frac{2}{x}$ $2x$

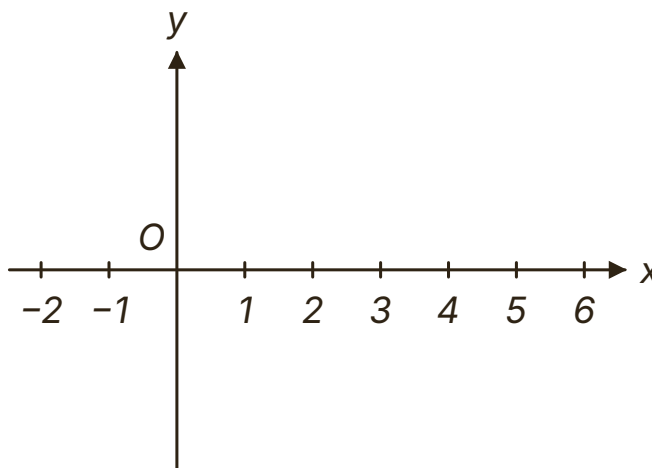
Answer

D9 $y = f(x)$ is a cubic function. [3 marks]

$f(x) < 0$ for $x < -1$ and $f(x) \geq 0$ for $x \geq -1$.

The function is increasing for $x < 1$, decreasing for $1 < x < 4$ and increasing for $x > 4$.

Draw a possible sketch of $y = f(x)$ for values of x from -2 to 6 .



E Extension

Stretch

No route is given. Spend five minutes on a problem before you look at a hint. The hints are on the last page of the worksheet, and you can use one at a time.

E1 Two tangents to the curve $y = 2x^2 + 8$ pass through the origin. Work out the equations of the two tangents. [4 marks]

Answer $y = \dots\dots$ and $y = \dots\dots$

E2 The curve $y = x^3 + px^2 + qx$ has stationary points where $x = -2$ and where $x = 4$. [5 marks]

Work out the values of p and q . Work out the distance between the two stationary points. Give your answer as a surd in its simplest form.

Answer $p = \dots\dots$, $q = \dots\dots$, distance $\dots\dots$

E3 Work out the shortest distance from the origin O to the curve $y = x^2 - 4$. [5 marks]
Give your answer in its simplest exact form.

Answer $\dots\dots$

F How did it go?

Colour one circle on each line.

	Not yet	Nearly	Yes
I can work out the gradient of a curve by differentiating, including negative powers of x and expressions I expand or divide first.	☐	☐	☐
I can find the equation of a tangent or a normal to a curve, and the points where tangents meet conditions.	☐	☐	☐
I can find where a function is increasing or decreasing, and use the second derivative.	☐	☐	☐
I can find stationary points, decide their nature, and solve problems about a greatest or least value.	☐	☐	☐

Extension hints

Use one hint at a time. Cover the ones below it with a sheet of paper.

-
- E1**
- Hint 1** Call the point of contact $(p, 2p^2 + 8)$. The tangent there has gradient $4p$.
- Hint 2** A line through the origin with gradient $4p$ is $y = 4px$. It passes through the point of contact, so $2p^2 + 8 = 4p \times p$.
- Hint 3** $2p^2 = 8$, so $p = 2$ or $p = -2$. The gradients are 8 and -8 : $y = 8x$ and $y = -8x$.
-
- E2**
- Hint 1** $\frac{dy}{dx} = 3x^2 + 2px + q$ is zero at $x = -2$ and $x = 4$, so it is $3(x + 2)(x - 4)$.
- Hint 2** $3(x + 2)(x - 4) = 3x^2 - 6x - 24$, so $2p = -6$ and $q = -24$. The points are $(-2, 28)$ and $(4, -80)$.
- Hint 3** The distance is $\sqrt{6^2 + 108^2} = \sqrt{11700} = \sqrt{900 \times 13} = 30\sqrt{13}$.
-
- E3**
- Hint 1** A point on the curve is $P(x, x^2 - 4)$. Work with $D = OP^2$: the shortest distance gives the least D .
- Hint 2** $D = x^2 + (x^2 - 4)^2 = x^4 - 7x^2 + 16$, and $\frac{dD}{dx} = 4x^3 - 14x = 2x(2x^2 - 7)$.
- Hint 3** $x = 0$ gives $D = 16$; $x^2 = \frac{7}{2}$ gives $D = \frac{15}{4}$, which is smaller. The shortest distance is $\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$.