



Mensuration, circle theorems, geometric proof and trigonometry

About 90 minutes Calculator allowed

113 marks

Date _____

A Do Now

The first one is what today needs. The others bring back earlier work.

A1 Needed today [2 marks]

Work out the exact value of $\sin 45^\circ \cos 45^\circ + \tan^2 30^\circ$.

Answer _____

A2 From chapter 4 [2 marks]

Solve $3y^2 + 5y - 2 = 0$.

Answer _____

A3 From chapter 1 [2 marks]

Write $\frac{10}{\sqrt{5}} + \sqrt{45}$ in the form $a\sqrt{5}$, where a is an integer.

Answer _____

A4 From chapter 5 [2 marks]

The point $P(-2, 6)$ lies on the circle $x^2 + y^2 = 40$. Work out the gradient of the tangent to the circle at P .

Answer _____

A5

GCSE Higher

[2 marks]

A sphere has a volume of $288\pi \text{ cm}^3$. Work out its radius.

(The volume of a sphere is $\frac{4}{3}\pi r^3$.)

Answer cm

B Examples, then your turn

Level 2

Your teacher works through each example with you. Then try the one beside it on your own.

EXAMPLE 1

Solve $4 \cos^2 x + 4 \sin x - 1 = 0$ for $0^\circ \leq x \leq 360^\circ$.

Hint Replace $\cos^2 x$ by $1 - \sin^2 x$:

$4 - 4 \sin^2 x + 4 \sin x - 1 = 0$, so

$4 \sin^2 x - 4 \sin x - 3 = 0$, which is

$(2 \sin x - 3)(2 \sin x + 1) = 0$. $\sin x = \frac{3}{2}$ has no solutions. $\sin x = -\frac{1}{2}$ gives $x = 210^\circ$ or $x = 330^\circ$.

B1 Solve

[4 marks]

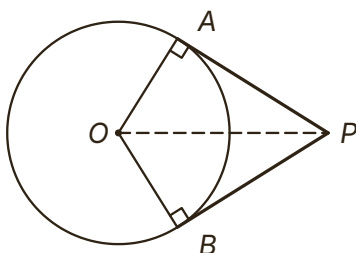
$3 \cos^2 x - \sin x - 1 = 0$ for $0^\circ \leq x \leq 360^\circ$. Give your answers to 1 decimal place where they are not exact.

Answer $x = \dots\dots$

EXAMPLE 2

PA and PB are tangents to the circle, centre O , at A and B .

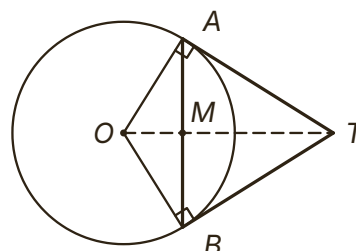
Prove that OP bisects angle APB .



Hint $OA = OB$, as they are radii. Angle $OAP =$ angle $OBP = 90^\circ$, as a tangent is perpendicular to the radius. OP is a side of both triangles. So triangles OAP and OBP are congruent (RHS), angle $AP O =$ angle $BP O$, and OP bisects angle APB .

B2 TA and TB are tangents to the circle, centre O , at A and B . The line OT meets the chord AB at M .

Prove that OT is perpendicular to AB .

**EXAMPLE 3**

Prove that $\frac{\sin x}{1 - \cos x} - \frac{1}{\tan x} \equiv \frac{1}{\sin x}$.

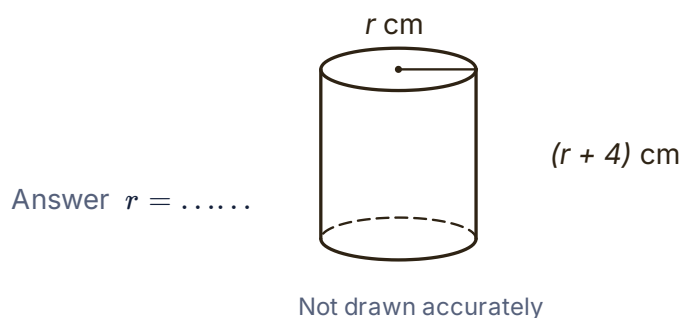
Hint Write $\frac{1}{\tan x}$ as $\frac{\cos x}{\sin x}$ and use the common denominator $\sin x(1 - \cos x)$. The top is $\sin^2 x - \cos x + \cos^2 x = 1 - \cos x$, which cancels with the $1 - \cos x$ underneath.

B3 Prove that $\frac{\cos x}{1 - \sin x} - \tan x \equiv \frac{1}{\cos x}$. [4 marks]

Each question asks for something different. Show your working. Give answers to 3 significant figures, and angles to 1 decimal place, unless the question says otherwise.

- C1** A solid cylinder has radius r cm and height $(r + 4)$ cm. Its total surface area is 140π cm². [3 marks]

Work out the value of r .



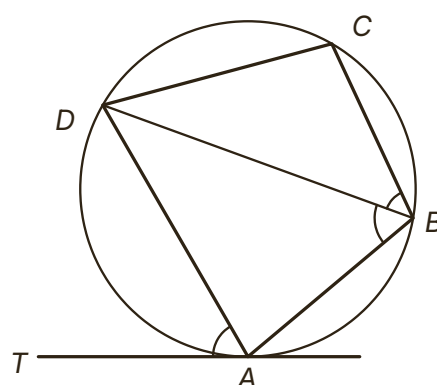
- C2** Each interior angle of a regular polygon is 7 times its exterior angle. Work out the number of sides of the polygon. [3 marks]

Answer

- C3** TA is a tangent to the circle at A . $ABCD$ is a cyclic quadrilateral. Angle $TAD = (x + 18)^\circ$, angle $ABD = (2x - 7)^\circ$ and angle $DBC = (x + 4)^\circ$. [4 marks]

Work out the size of angle ADC . Give reasons.

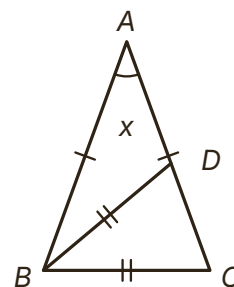
Answer angle $ADC = \dots\dots^\circ$



Not drawn accurately

- C4** In triangle ABC , $AB = AC$. D is the point on AC such that $BD = BC$. Angle $BAC = x$. [3 marks]

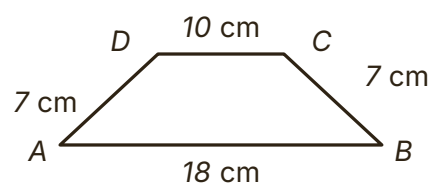
Prove that angle $ABD = 90^\circ - \frac{3x}{2}$.



Not drawn accurately

- C5** $ABCD$ is an isosceles trapezium. AB is parallel to DC . $AB = 18$ cm, $DC = 10$ cm and $AD = BC = 7$ cm.

- (a) Work out the size of angle DAB . [2 marks]



Answer°

Not drawn accurately

- (b) Work out the area of the trapezium. [2 marks]

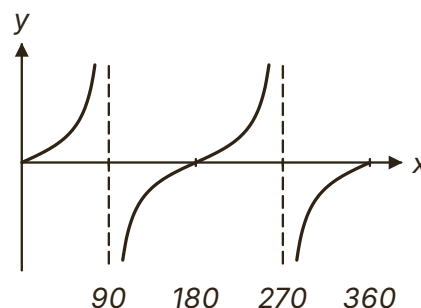
Answer cm²

Total for C5: 4 marks

- C6** Here is the graph of $y = \tan x$ for $0^\circ \leq x \leq 360^\circ$. [2 marks]

You are given that $\tan 65^\circ = t$. Work out the two values of x , for $0^\circ \leq x \leq 360^\circ$, for which $\tan x = -t$.

Answer $x = \dots\dots$



C7 Solve $(3 \tan x - 1)(5 \sin x + 4) = 0$ for $0^\circ \leq x \leq 360^\circ$.

[4 marks]

Answer $x = \dots\dots$

C8 Show that $\frac{\sin^2 x - 1}{\sin x \cos x} \times \tan x$ simplifies to an integer.

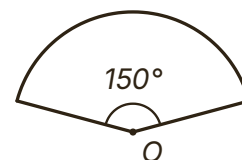
[3 marks]

Answer $\dots\dots$

C9 A sector of a circle, centre O , has an angle of 150° . Its area is $15\pi \text{ cm}^2$.

[4 marks]

Work out the perimeter of the sector. Give your answer in the form $a + b\pi$.



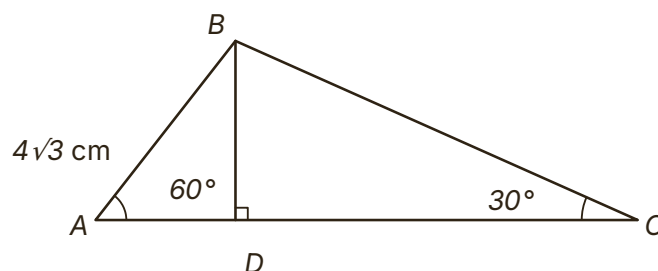
Answer $\dots\dots \text{ cm}$

Not drawn accurately

Answer every question in the space given. Show your working: most of the marks are for method.

- D1** A , D and C lie on a straight line. BD is perpendicular to AC . $AB = 4\sqrt{3}$ cm, angle $BAD = 60^\circ$ and angle $BCD = 30^\circ$.

- (a) Show that $BC = 12$ cm. You must show your working. [3 marks]



Not drawn accurately

- (b) Work out the exact length of AC . Give your answer in the form $a\sqrt{3}$. [3 marks]

Answer $AC = \dots\dots$ cm

Total for D1: 6 marks

- D2** Ben solves $\tan x = -1.2$ for $0^\circ \leq x \leq 360^\circ$. He writes, " $x = -50.2^\circ$ and $x = 129.8^\circ$." [3 marks]

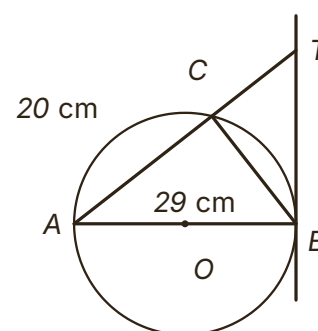
Explain what Ben has done wrong, and give the correct solutions.

Answer $x = \dots\dots$

- D3** A , B and C are points on a circle, centre O . AB is a diameter of the circle. $AB = 29$ cm and $AC = 20$ cm. The tangent to the circle at B meets AC extended at T .

- (a) Write down the size of angle ACB . Give a reason. [1 mark]

Answer $\dots\dots^\circ$



Not drawn accurately

(b) Work out the length of BC .

[2 marks]

Answer $BC = \dots\dots$ cm

(c) Work out the length of BT .

[3 marks]

Answer $BT = \dots\dots$ cm

Total for D3: 6 marks

D4 Answer all three parts.

(a) Show that the equation $2 \sin^2 x - \cos x - 1 = 0$ can be written as
 $2 \cos^2 x + \cos x - 1 = 0$.

[2 marks]

(b) Hence solve $2 \sin^2 x - \cos x - 1 = 0$ for $0^\circ \leq x \leq 360^\circ$.

[3 marks]

Answer $x = \dots\dots$

(c) Hence write down the solutions of $2 \sin^2 x - \cos x - 1 = 0$ for
 $360^\circ \leq x \leq 720^\circ$.

[1 mark]

Answer $x = \dots\dots$

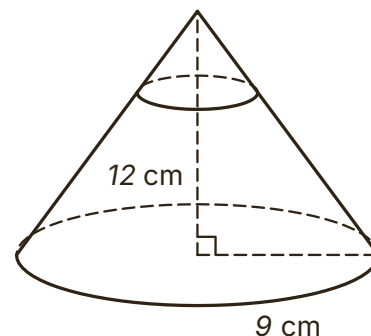
Total for D4: 6 marks

- D5** A solid cone has base radius 9 cm and vertical height 12 cm. The top of the cone is cut off by a cut parallel to the base, 4 cm below the vertex. The part left is a frustum.

(Curved surface area of a cone = $\pi r l$. Volume of a cone = $\frac{1}{3}\pi r^2 h$.)

- (a) Show that the slant height of the whole cone is 15 cm. [1 mark]

- (b) Work out the curved surface area of the frustum. Give your answer as a multiple of π . [3 marks]



Answer cm^2

- (c) Work out the volume of the frustum. Give your answer as a multiple of π . [2 marks]

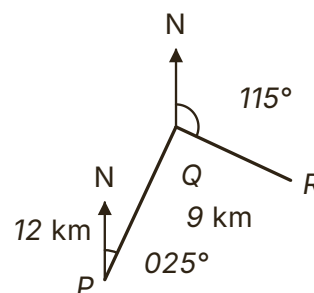
Answer cm^3

Total for D5: 6 marks

- D6** A boat sails 12 km from P to Q on a bearing of 025° . It then sails 9 km from Q to R on a bearing of 115° .

- (a) Show that angle $PQR = 90^\circ$. [2 marks]

- (b) Work out the distance PR . [2 marks]



Not drawn accurately

Answer $PR = \dots\dots$ km

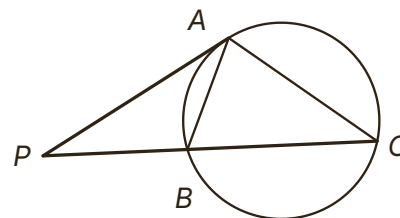
- (c) Work out the bearing of R from P . Give your answer to the nearest degree. [2 marks]

Answer $^\circ$

Total for D6: 6 marks

D7 PA is a tangent to the circle at A . B and C are points on the circle, and PBC is a straight line.

- (a) Prove that triangles PAB and PCA are similar. [3 marks]



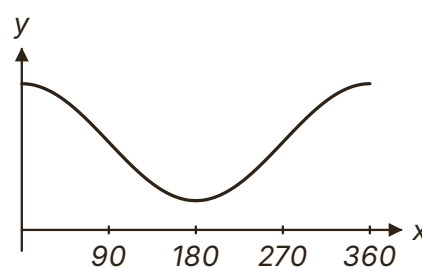
- (b) $PB = 4$ cm and $BC = 5$ cm. Hence work out the length of PA . [2 marks]

Answer $PA = \dots\dots$ cm

Total for D7: 5 marks

D8 Here is the graph of $y = 3 + 2 \cos x$ for $0^\circ \leq x \leq 360^\circ$.

- (a) Write down the greatest value of y and the coordinates of the minimum point. [2 marks]



Answer $\dots\dots$; $(\dots\dots, \dots\dots)$

- (b) Solve $3 + 2 \cos x = 2.4$ for $0^\circ \leq x \leq 360^\circ$. [3 marks]

Answer $x = \dots\dots$

Total for D8: 5 marks

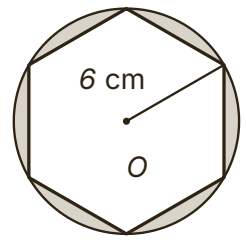
D9 Which of these is equal to $\tan 335^\circ$? Circle your answer. [1 mark]

$\tan 25^\circ$ $-\tan 25^\circ$ $\tan 65^\circ$ $-\tan 155^\circ$

Answer

- D10** A regular hexagon is drawn inside a circle of radius 6 cm, centre O , with its six vertices on the circle. [4 marks]

Show that the area of the region inside the circle but outside the hexagon is $(36\pi - 54\sqrt{3}) \text{ cm}^2$. You must show your working.



E Extension

Stretch

No route is given. Spend five minutes on a problem before you look at a hint. The hints are on the last page of the worksheet, and you can use one at a time.

- E1** Solve $2 \sin x \tan x = 3$ for $0^\circ \leq x \leq 360^\circ$. [4 marks]

Answer $x = \dots\dots$

- E2** x is an acute angle and $\sin x + \cos x = \frac{7}{5}$. [5 marks]

Work out the two possible values of $\tan x$.

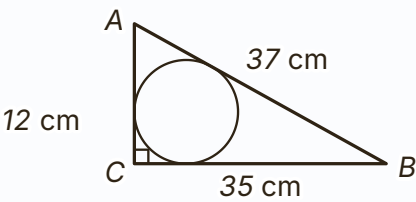
Answer $\tan x = \dots\dots$ or $\dots\dots$

E3

ABC is a right-angled triangle with $AC = 12\text{ cm}$, $BC = 35\text{ cm}$ and $AB = 37\text{ cm}$. A circle is drawn inside the triangle so that it touches all three sides.

[4 marks]

Work out the radius of the circle.



Answer cm Not drawn accurately

F How did it go?

Colour one circle on each line.

	Not yet	Nearly	Yes
I can find lengths, areas and volumes with algebra, including sectors, cones, frustums and polygons.	☐	☐	☐
I can use the circle theorems and give the reason for each step.	☐	☐	☐
I can write a geometric proof using angle facts, circle theorems and similar triangles.	☐	☐	☐
I can use trigonometry in right-angled triangles, with exact values for 30, 45 and 60 degrees.	☐	☐	☐
I can use the graphs and symmetry of sine, cosine and tangent for any angle.	☐	☐	☐
I can solve trigonometric equations, including quadratics in sine or cosine.	☐	☐	☐
I can prove identities and solve equations using the two trigonometric identities.	☐	☐	☐

Extension hints

Use one hint at a time. Cover the ones below it with a sheet of paper.

-
- E1** **Hint 1** Write $\tan x$ as $\frac{\sin x}{\cos x}$ and multiply both sides by $\cos x$.
- Hint 2** $2 \sin^2 x = 3 \cos x$, so $2 - 2 \cos^2 x = 3 \cos x$, which is
 $2 \cos^2 x + 3 \cos x - 2 = 0$.
- Hint 3** $(2 \cos x - 1)(\cos x + 2) = 0$. $\cos x = -2$ is impossible, so $\cos x = \frac{1}{2}$: $x = 60^\circ$
 or 300° .
-
- E2** **Hint 1** Square both sides and use $\sin^2 x + \cos^2 x = 1$.
- Hint 2** $1 + 2 \sin x \cos x = \frac{49}{25}$, so $\sin x \cos x = \frac{12}{25}$. Now you know the sum and the
 product of $\sin x$ and $\cos x$.
- Hint 3** They are the roots of $t^2 - \frac{7}{5}t + \frac{12}{25} = 0$, that is $(5t - 3)(5t - 4) = 0$. Either
 $\sin x = \frac{3}{5}$ and $\cos x = \frac{4}{5}$, or the other way round.
-
- E3** **Hint 1** The radii to the two sides at the right angle make a square of side r with the
 corner C .
- Hint 2** The two tangents from a point to a circle are equal. From A both are $12 - r$;
 from B both are $35 - r$.
- Hint 3** AB is made of one tangent from A and one from B :
 $(12 - r) + (35 - r) = 37$, so $r = 5$.
-