



Multiplying matrices, transformations and combining them

About 80 minutes No calculator 106 marks

Date _____

A Do Now

The first is what this chapter needs. The rest bring back earlier work so nothing is forgotten.

A1 Needed today [2 marks]

Write down the image of the point $(3, -6)$ after (a) a reflection in the line $y = x$
(b) a rotation of 180° about O .

Answer (a) $(\dots, \dots)$ (b) $(\dots, \dots)$

A2 From chapter 8 [2 marks]

$y = 2x^3 - 5x^2 + 4$. Work out the gradient of the curve at the point where $x = 2$.

Answer

A3 From chapter 6 [2 marks]

Solve $2 \cos x = -1$ for $0^\circ \leq x \leq 360^\circ$.

Answer $x = \dots\dots$ or $x = \dots\dots$

A4 From chapter 4 [2 marks]

Solve the simultaneous equations $3x + 2y = 4$ and $x - y = 3$.

Answer $x = \dots\dots$ $y = \dots\dots$

A5

GCSE Higher

[2 marks]

Expand and simplify $(2x - 3)(x + 4) - x(x + 5)$.

Answer

B Examples, then your turn

Level 2

Your teacher works through each example with you. Then try the one beside it on your own.

EXAMPLE 1

$$\mathbf{M} = \begin{pmatrix} a & 3 \\ 2 & b \end{pmatrix} \text{ and } \mathbf{M}^2 = \begin{pmatrix} 10 & 9 \\ 6 & 7 \end{pmatrix}.$$

Work out the values of a and b .

Hint Multiply $\mathbf{M}$ by itself and match each entry. Use the two equations with squares, then the one that decides the signs.

B1

$$\mathbf{N} = \begin{pmatrix} p & -2 \\ 3 & q \end{pmatrix} \text{ and } \mathbf{N}^2 = \begin{pmatrix} -5 & -4 \\ 6 & 3 \end{pmatrix}.$$

[4 marks]

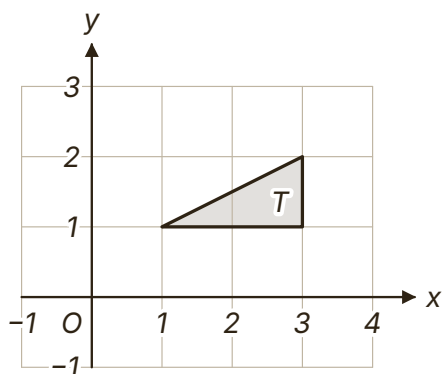
Work out the values of p and q .

Answer $p = \dots\dots$ $q = \dots\dots$

EXAMPLE 2

Triangle T is reflected in the x -axis. The image is then reflected in the line $y = x$.

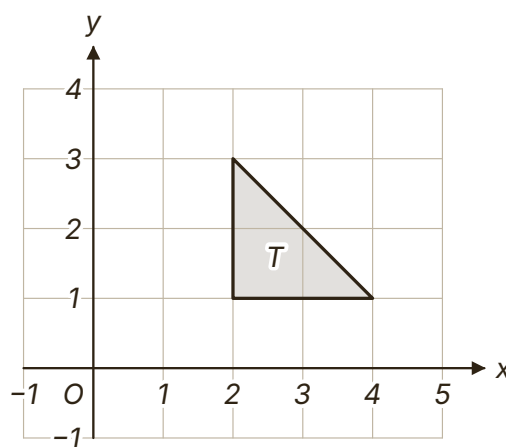
- Work out the matrix that represents the combined transformation.
- Describe fully the single transformation it represents.
- Work out the image of the vertex $(3, 2)$.



Hint The first transformation is on the right: (reflection in $y = x$) $\times$ (reflection in the x -axis).

B2 Triangle T is reflected in the y -axis. The image is then rotated 90° anticlockwise about O . [4 marks]

- Work out the matrix that represents the combined transformation.
- Describe fully the single transformation it represents.
- Work out the image of the vertex $(4, 1)$.



Answer (a) $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ (c) $(\dots, \dots)$

EXAMPLE 3

The matrix $\mathbf{M}$ maps the point $(2, 1)$ to $(4, -1)$ and the point $(1, 3)$ to $(7, 2)$.

Work out $\mathbf{M}$.

Hint Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Each point gives one equation for the top row and one for the bottom row.

B3 The matrix $\mathbf{M}$ maps the point $(1, 2)$ to $(4, 5)$ and the point $(3, 1)$ to $(7, 0)$. [4 marks]

Work out $\mathbf{M}$.

Answer $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

EXAMPLE 4

$\mathbf{A} = \begin{pmatrix} k & 2 \\ 1 & -1 \end{pmatrix}$. The point $(2, 1)$ is transformed by $\mathbf{A}$, and its image is transformed by $\mathbf{A}$ again. The final image is $(26, 7)$.

Work out the value of k .

Hint Work out the first image, then the second, in terms of k . One coordinate gives a quadratic with two roots; the other coordinate decides.

B4 $\mathbf{A} = \begin{pmatrix} k & -1 \\ 2 & 1 \end{pmatrix}$. The point $(1, 3)$ is transformed by $\mathbf{A}$, and its image is transformed by $\mathbf{A}$ again. The final image is $(5, 9)$. [4 marks]

Work out the value of k .

Answer $k = \dots\dots$

Each question asks for something different. Show your working.

C1 $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}$. Work out $\mathbf{AB}$ and $\mathbf{BA}$. [3 marks]

Answer $\mathbf{AB} = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$

C2 $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$. Show that $\mathbf{A}^2 = \mathbf{I}$. Hence write down $\mathbf{A}^5$. [3 marks]

Answer $\mathbf{A}^5 = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$

C3 $\begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$. Work out the two possible pairs of values of a and b . [3 marks]

Answer $a = \dots$, $b = \dots$ or $a = \dots$, $b = \dots$

C4 Write down the matrix that represents (a) a reflection in the line $y = -x$ (b) an enlargement, scale factor 5, centre O . [2 marks]

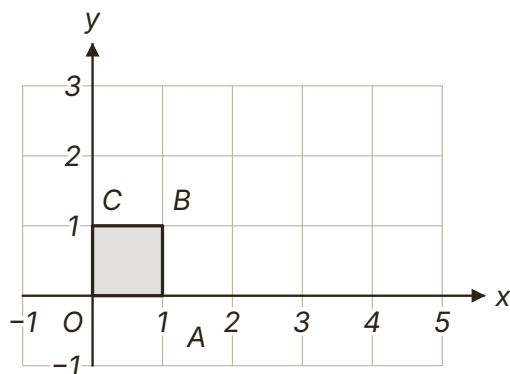
Answer (a) $\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$ (b) $\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$

C5 Describe fully the single transformation represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. [2 marks]

- C6** The unit square $OABC$ is transformed by the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. [3 marks]

Draw the image of the unit square on the grid, and work out its area.

Answer area =



- C7** The matrix $\begin{pmatrix} k & 3 \\ -2 & k \end{pmatrix}$ maps the point $(2, -1)$ to the point $(5, -8)$. Work out the value of k . [2 marks]

Answer $k = \dots\dots$

- C8** $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about O . [2 marks]
- Work out $\mathbf{R}^3$. Describe fully the transformation it represents.

Answer $\mathbf{R}^3 = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$

- C9** The matrix $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ maps the point P to the point $(7, 5)$. Work out the coordinates of P . [2 marks]

Answer $(\dots\dots, \dots\dots)$

D Exam-style questions

AQA exam style

Answer every question in the space given. Show your working: most of the marks are for method.

D1 $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

(a) Work out $\mathbf{AB}$.

[2 marks]

Answer $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

(b) The point Q is the image of a point under $\mathbf{A}$. Hence explain what $\mathbf{B}$ does to Q . [1 mark]

(c) The point P is mapped to $(8, 5)$ by $\mathbf{A}$. Use $\mathbf{B}$ to work out the coordinates of P . [2 marks]

Answer $(\dots, \dots)$

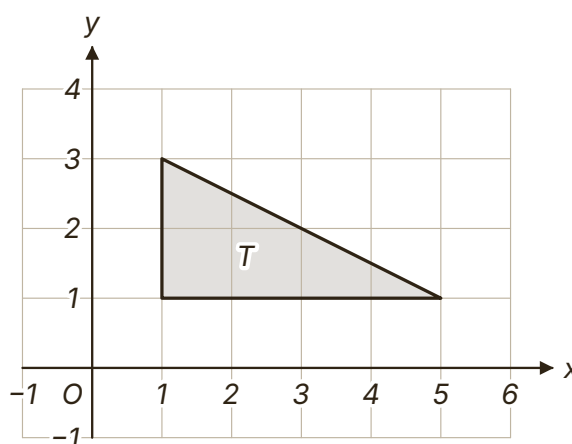
Total for D1: 5 marks

D2 Triangle T is reflected in the line $y = -x$ to give T_1 . T_1 is then enlarged, scale factor 3, centre O , to give T_2 .

(a) Write down the matrix $\mathbf{P}$ for the reflection and the matrix $\mathbf{Q}$ for the enlargement. [2 marks]

Answer $\mathbf{P} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

(b) Work out the single matrix that maps T to T_2 . [2 marks]

Answer $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

(c) Work out the image in T_2 of the vertex $(5, 1)$.

[1 mark]

Answer $(\dots, \dots)$

(d) Work out the area of T_2 .

[1 mark]

Answer $\dots$ square units

Total for D2: 6 marks

D3 Ben says, "A rotation of 180° about O followed by an enlargement, scale factor -2 , centre O , is an enlargement with scale factor -2 ." [3 marks]

Ben is wrong. Use matrices to show that he is wrong, and describe fully the single transformation.

D4 $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}.$

(a) Given that $\mathbf{M}^2 = p\mathbf{M} + q\mathbf{I}$, work out the values of p and q . [3 marks]

Answer $p = \dots$ $q = \dots$

(b) Hence write $\mathbf{M}^3$ in the form $a\mathbf{M} + b\mathbf{I}$. [2 marks]

Answer $\mathbf{M}^3 = \dots\mathbf{M} + \dots\mathbf{I}$

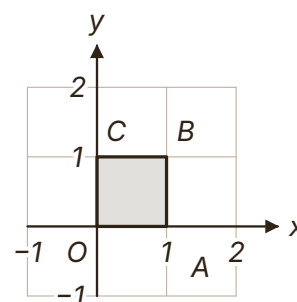
Total for D4: 5 marks

D5 $\mathbf{M} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$, where $k > 0$. The unit square $OABC$ is transformed by $\mathbf{M}$. The area of its image is 49.

(a) Work out the value of k . [2 marks]

Answer $k = \dots\dots$

(b) $\mathbf{M}$ represents a combination of two transformations. Describe them fully. [2 marks]



(c) Work out the image of the point $(-2, 3)$. [1 mark]

Answer $(\dots\dots, \dots\dots)$

Total for D5: 5 marks

D6 The matrix $\begin{pmatrix} p & q \\ q & p \end{pmatrix}$ maps the point $(2, 5)$ to the point $(11, 17)$.

(a) Work out the values of p and q . [2 marks]

Answer $p = \dots\dots$ $q = \dots\dots$

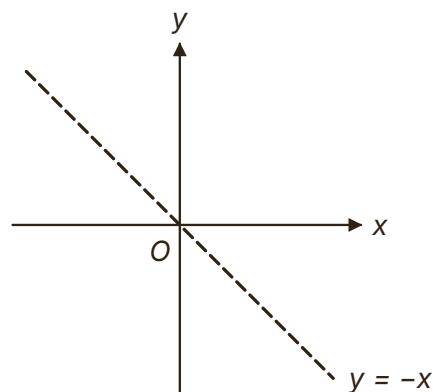
(b) Hence work out the coordinates of the point that is mapped to $(8, 8)$. [2 marks]

Answer $(\dots\dots, \dots\dots)$

Total for D6: 4 marks

D7 A shape is reflected in the y -axis, and the image is then reflected in the line $y = -x$.

- (a) Work out the matrix for the combined transformation, and describe fully the single transformation it represents. [3 marks]



- (b) The two reflections are now done in the other order. Describe fully the single transformation. [2 marks]

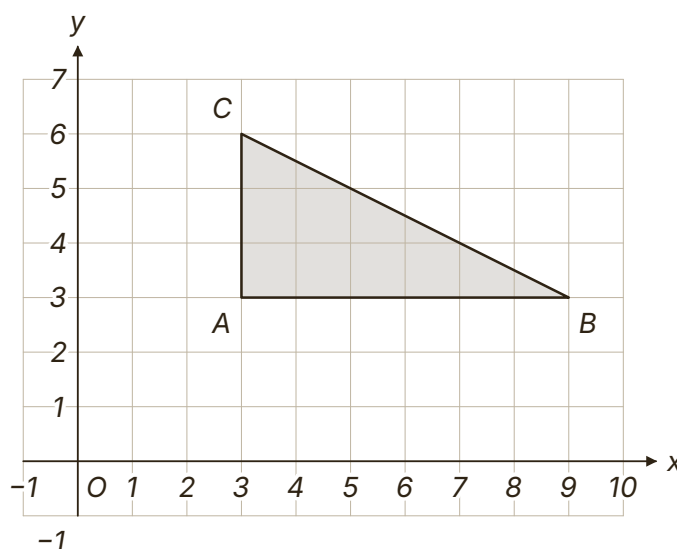
Total for D7: 5 marks

D8 Triangle ABC has vertices $A(3, 3)$, $B(9, 3)$ and $C(3, 6)$. It is transformed by the matrix $\begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$.

- (a) Work out the coordinates of the image of B . [1 mark]

Answer (.....,)

- (b) Describe fully the transformation the matrix represents. [1 mark]



- (c) Work out the area of the image of triangle ABC . [2 marks]

Answer square units

Total for D8: 4 marks

D9 $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

(a) Work out $\mathbf{A}^2$ and $\mathbf{A}^3$.

[2 marks]

Answer $\mathbf{A}^2 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ $\mathbf{A}^3 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

(b) Hence write down $\mathbf{A}^n$ in terms of n .

[1 mark]

Answer $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

(c) $\mathbf{A}^n$ maps the point $(1, 1)$ to the point $(61, 1)$. Work out the value of n .

[2 marks]

Answer $n = \dots\dots$

Total for D9: 5 marks

E Extension

Stretch

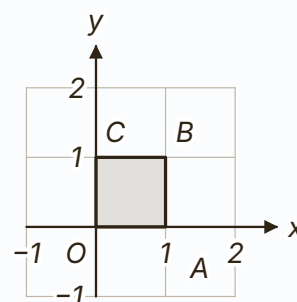
No route is given. Spend five minutes on a problem before you look at a hint. The hints are on the last page of the worksheet, and you can use one at a time.

E1 $\mathbf{M} = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}$, where $k > 1$. The unit square $OABC$ is transformed by $\mathbf{M}$. [4 marks]

The area of its image is 8.

Work out the value of k .

Answer $k = \dots\dots$



- E2** $\mathbf{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. $\mathbf{A}^2 = \mathbf{I}$, and $\mathbf{A}$ maps the point $(1, 1)$ to the point $(5, -3)$. [4 marks]
Work out $\mathbf{A}$.

Answer $\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

- E3** T is a transformation. A reflection in the line $y = x$ followed by T is the same as an enlargement, scale factor -5 , centre O . [4 marks]
Work out the matrix for T . Describe T fully as a combination of two transformations.

Answer $\mathbf{T} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

- E4** $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. [4 marks]
Show that $\mathbf{M}^6 = \mathbf{I}$. Hence work out $\mathbf{M}^{100}$.

Answer $\mathbf{M}^{100} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$

F How did it go?

Colour one circle on each line.

	Not yet	Nearly	Yes
I can multiply matrices, use the identity matrix, and find unknown entries from a product or a power.	☐	☐	☐
I can transform points and the unit square with a matrix, find the matrix for a transformation, and describe what a matrix does.	☐	☐	☐
I can combine transformations by multiplying their matrices in the right order, and describe the single transformation.	☐	☐	☐

Extension hints

Use one hint at a time. Cover the ones below it with a sheet of paper.

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- E1** **Hint 1** Work out the images of $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$. The image is a parallelogram.
- Hint 2** The images are $(1, k)$, $(1 + k, 1 + k)$ and $(k, 1)$. Draw the square from $(0, 0)$ to $(1 + k, 1 + k)$ round it and take away the two regions outside the parallelogram.
- Hint 3** Each region outside splits into two triangles of area $\frac{1}{2}(1 + k)$, so the parallelogram has area $(1 + k)^2 - 2(1 + k) = k^2 - 1$. Then $k^2 - 1 = 8$, so $k = 3$.
-
- E2** **Hint 1** Multiply **A** by itself. The top-left entry gives $a^2 + bc = 1$.
- Hint 2** The image of $(1, 1)$ gives $a + b = 5$ and $c - a = -3$, so $b = 5 - a$ and $c = a - 3$.
- Hint 3** Substitute: $a^2 + (5 - a)(a - 3) = 1$, which is $8a - 15 = 1$, so $a = 2$, $b = 3$, $c = -1$.
-
- E3** **Hint 1** Let **Q** be the matrix for the reflection. Then $\mathbf{TQ} = -5\mathbf{I}$.
- Hint 2** $\mathbf{Q}^2 = \mathbf{I}$, because reflecting twice puts every point back. Multiply both sides on the right by **Q**: $\mathbf{T} = -5\mathbf{Q}$.
- Hint 3** $\mathbf{T} = \begin{pmatrix} 0 & -5 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.
-
- E4** **Hint 1** Work out $\mathbf{M}^2$, then $\mathbf{M}^3 = \mathbf{MM}^2$.
- Hint 2** $\mathbf{M}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$, so $\mathbf{M}^6 = (-\mathbf{I})^2 = \mathbf{I}$.
- Hint 3** $100 = 6 \times 16 + 4$, so $\mathbf{M}^{100} = \mathbf{M}^4 = \mathbf{M}^3\mathbf{M} = -\mathbf{M}$.
-