Proof by induction: divisibility

96 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-induction-divisibility/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Show that $3 \times 2^k + 2^{k+1} = 5 \times 2^k$.

Answer .....

A2 Needed today [2 marks]

Show that $7^{k+1} - 7^k$ is a multiple of 6.

Answer .....

A3 Last lesson [2 marks]

An induction proof has shown that a result is true for $n = 1$, and that if it is true for $n = k$ then it is true for $n = k + 1$. Write the conclusion.

Answer .....

A4 From A-Level Maths [2 marks]

Factorise $n^3 - n$ fully.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 One power of a number

EXAMPLE 1 · I DO

Prove by induction that $5^n - 1$ is divisible by 4 for all positive integers n.

Hint Let $f(n) = 5^n - 1$ and assume $f(k) = 4A$. Write 5^{k+1} as 5×5^k, and 5^k as $4A + 1$.

YOUR TURN · WE DO

B1 Prove by induction that $11^n - 1$ is divisible by 10 for all positive integers n. [5 marks]

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EXAMPLE 2 · I DO

Prove by induction that $3^{2n} + 7$ is divisible by 8 for all positive integers n.

Hint Work out $f(k+1) - f(k)$. If $f(k)$ and the difference are both multiples of 8, so is $f(k+1)$.

YOUR TURN · WE DO

B2 Prove by induction that $6^n + 4$ is divisible by 5 for all positive integers n. [5 marks]

PRACTICE · YOU DO

C1 Prove by induction that $13^n - 1$ is divisible by 12 for all positive integers n. [4 marks]

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C2 The result is that $5 \times 9^n + 3$ is divisible by 8 for all positive integers n. Sam starts the step: "$f(k+1) = 5 \times 9^{k+1} + 3 = 45^{k+1} + 3$." Explain Sam's mistake, and prove the result by induction. [6 marks]

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PART 2 Two powers of n

EXAMPLE 3 · I DO

Prove by induction that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers n.

Hint Write $f(k+1) = 2 \times 2^{k+2} + 9 \times 3^{2k+1}$. Split the 9 as $2 + 7$, so that part of it is $2f(k)$.

YOUR TURN · WE DO

B3 Prove by induction that $4^{n+1} + 5^{2n-1}$ is divisible by 21 for all positive integers n. [6 marks]

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EXAMPLE 4 · I DO

Prove by induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.

Hint Expand $(k+1)^3 + 2(k+1)$ and find $f(k)$ inside it.

YOUR TURN · WE DO

B4 Prove by induction that $n^3 + 5n$ is divisible by 6 for all positive integers n. [6 marks]

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C3 In this question you must show detailed reasoning. Prove by induction that $2n^3 + 3n^2 + n$ is divisible by 6 for all positive integers n. [5 marks]

C4 Let $f(n) = 12^n + 2 \times 5^{n-1}$.

(a) Show that $f(k+1) - 5f(k) = 7 \times 12^k$. [3 marks]

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(b) Hence prove by induction that $f(n)$ is divisible by 7 for all positive integers n. [5 marks]

Total for C4: 8 marks

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PART 3 Starting from zero

EXAMPLE 5 · I DO

Prove by induction that $4^n + 6n - 1$ is divisible by 9 for all non-negative integers n.

Hint The base case is $n = 0$. Write $f(k+1) = 4 \times 4^k + 6k + 5$, and take $4f(k)$ out of it.

YOUR TURN · WE DO

B5 Prove by induction that $7^n + 3n - 1$ is divisible by 9 for all non-negative integers n. [6 marks]

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C5 Prove by induction that $3^n + 2n - 1$ is divisible by 4 for all non-negative integers n. [5 marks]

C6 Prove by induction that $2^{2n+1} + 3^{2n+1}$ is divisible by 5 for all non-negative integers n. [6 marks]

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D1 Let $f(n) = 3^{2n} + 8n - 1$.

(a) Show that $f(k+1) - 9f(k) = 16(1 - 4k)$. [3 marks]

(b) Hence prove by induction that $f(n)$ is divisible by 16 for all positive integers n. [6 marks]

Total for D1: 9 marks

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D2

> Kai writes: "If $f(k)$ is divisible by 4 then so is $f(k+1)$. So $5^n + 2$ is divisible by 4 for every positive integer n."

Let $f(n) = 5^n + 2$.

(a) Show that if $f(k)$ is divisible by 4 then $f(k+1)$ is divisible by 4. [3 marks]

(b) Explain why Kai's conclusion is wrong. [2 marks]

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(c) Prove by induction that $9^n + 7$ is divisible by 8 for all positive integers n. [4 marks]

Total for D2: 9 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 Prove by induction that $3^{2^n} - 1$ is divisible by 2^{n+2} for all positive integers n. [8 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Check $n = 1$, then assume $3^{2^k} - 1 = 2^{k+2}A$.

Hint 2 $2^{k+1} = 2 \times 2^k$, so $3^{2^{k+1}}$ is the square of 3^{2^k}. Use the difference of two squares.

Hint 3 You need one more factor of 2 than you assumed. Look at $3^{2^k} + 1$: is it odd or even?

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can prove a divisibility result by induction, writing $f(k) = mA$ for the assumption.	○ ○ ○
I can write $f(k+1)$ as a multiple of $f(k)$ plus a multiple of m.	○ ○ ○
I can use $f(k+1) - f(k)$, and say why $f(k+1)$ is then divisible.	○ ○ ○
I can handle a base case of $n = 0$, and a step of two for odd n.	○ ○ ○

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