Proof by induction: matrix powers and inequalities

93 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-induction-matrices/

Date

A Do Now

Quick questions to start.

A1 From chapter 3 [2 marks]

Work out $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$.

Answer .....

A2 From chapter 3 [2 marks]

Find the inverse of $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

Answer .....

A3 Needed today [2 marks]

Show that $k^2 - 2k - 1 > 0$ for every integer $k \geqslant 3$.

Answer .....

A4 Last lesson [2 marks]

Write $(k+1)!$ in terms of $k!$, and 3^{k+1} in terms of 3^k.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 Powers of a matrix

EXAMPLE 1 · I DO

Prove by induction that $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 3n \\ 0 & 1 \end{pmatrix}$ for all positive integers n.

Hint Check $n = 1$. Assume it for $n = k$, then write the $(k+1)$th power as the kth power times the matrix, and multiply.

YOUR TURN · WE DO

B1 Prove by induction that $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ for all positive integers n. [5 marks]

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EXAMPLE 2 · I DO

Prove by induction that $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for all positive integers n.

Hint In the step, $2 \times 2^k = 2^{k+1}$. Look at the top right entry carefully.

YOUR TURN · WE DO

B2 Prove by induction that $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{pmatrix}$ for all positive integers n. [5 marks]

PRACTICE · YOU DO

C1 Prove by induction that $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -3n & 1 \end{pmatrix}$ for all positive integers n. [4 marks]

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C2 Let $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. Sam tries to prove $\mathbf{M}^n = \begin{pmatrix} 1 & 4n \\ 0 & 1 \end{pmatrix}$ and writes in the step: "$\mathbf{M}^{k+1} = \mathbf{M}^k + \mathbf{M} = \begin{pmatrix} 2 & 4k+4 \\ 0 & 2 \end{pmatrix}$." Explain Sam's mistake, and prove the result by induction for all positive integers n. [6 marks]

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PART 2 Finding the formula for a power

EXAMPLE 3 · I DO

Prove by induction that $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}^n = \begin{pmatrix} 1+2n & -n \\ 4n & 1-2n \end{pmatrix}$ for all positive integers n.

Hint Multiply out all four entries in the step, then factor each to show $k+1$ in place of n.

YOUR TURN · WE DO

B3 Prove by induction that $\begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & -4n \\ 4n & 1-4n \end{pmatrix}$ for all positive integers n. [6 marks]

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C3 [5 marks]

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$ for all positive integers n.

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C4 Let $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Work out $\mathbf{B}^2$ and $\mathbf{B}^3$, write a formula for $\mathbf{B}^n$, and prove it by induction for all positive integers n. [7 marks]

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PART 3 Inequalities

EXAMPLE 4 · I DO

Prove by induction that $2^n > 2n + 1$ for all integers $n \geqslant 3$.

Hint The base case is $n = 3$. In the step, $2^{k+1} = 2 \times 2^k > 2(2k + 1)$. Then show this is at least $2(k + 1) + 1$.

YOUR TURN · WE DO

B4 Prove by induction that $3^n > 4n + 2$ for all integers $n \geqslant 3$. [5 marks]

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EXAMPLE 5 · I DO

Prove by induction that $n! > 2^n$ for all integers $n \geqslant 4$.

Hint $(k+1)! = (k+1)k! > (k+1)2^k$. Then use $k+1 > 2$.

YOUR TURN · WE DO

B5 Prove by induction that $n! > 3^n$ for all integers $n \geqslant 7$. [5 marks]

PRACTICE · YOU DO

C5 Prove by induction that $4^n > 3n + 5$ for all integers $n \geqslant 2$. [5 marks]

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C6 Alex is proving that $5^n > 8n$ for all integers $n \geqslant 2$, and writes: "$n = 2$: $25 > 16$. $5^{k+1} = 5 \times 5^k > 8(k+1)$. So it is true for all $n \geqslant 2$." Explain what is missing from Alex's proof, and write a complete proof. [6 marks]

D Exam-style questions

OCR exam style

D1 Let $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Work out $\mathbf{M}^2$ and $\mathbf{M}^3$. [2 marks]

Answer .....

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(b) Prove by induction that $\mathbf{M}^n = \begin{pmatrix} 2^n & 3^n - 2^n \\ 0 & 3^n \end{pmatrix}$ for all positive integers n. [5 marks]

(c) In this question you must show detailed reasoning. Hence find the smallest positive integer n for which the top right entry of $\mathbf{M}^n$ is greater than 1000. [2 marks]

Answer .....

Total for D1: 9 marks

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D2

Mo writes: "$2^n > n^3$ for every positive integer n, because powers of 2 grow faster than cubes."

(a) Show that Mo is wrong. [2 marks]

(b) Find the smallest integer N with $N \geqslant 2$ such that $2^N > N^3$. [2 marks]

Answer .....

(c) Prove by induction that $2^n > n^3$ for all integers $n \geqslant N$. [5 marks]

Total for D2: 9 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The Fibonacci numbers are $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Let $\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Prove by induction that $\mathbf{Q}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all positive integers n. Hence show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. [8 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Check $n = 1$: what are F_2 and F_0?

Hint 2 In the step, multiply $\mathbf{Q}^k$ by $\mathbf{Q}$ and use $F_{k+1} + F_k = F_{k+2}$.

Hint 3 For the last part, take the determinant of both sides. The determinant of a product is the product of the determinants.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can prove a formula for $\mathbf{M}^n$ by writing $\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$.	○ ○ ○
I can work out small powers of a matrix and use them to make a conjecture.	○ ○ ○
I can prove an inequality by induction, showing why the step keeps the inequality.	○ ○ ○
I can use a proved result, for example to find the smallest n that works.	○ ○ ○

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