Proof by induction: sums

91 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-induction-sums/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Show that $\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.

Answer .....

A2 Needed today [1 mark]

The rth term of a series is $r(r+2)$. Write down the $(k+1)$th term, in factorised form.

Answer .....

A3 Needed today [3 marks]

Write $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ as a single fraction in its simplest form.

Answer .....

A4 From A-Level Maths [2 marks]

Write $(k+2)!$ in terms of $k!$.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 The four steps of a proof by induction

EXAMPLE 1 · I DO

Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for all positive integers n.

Hint Four steps: check $n = 1$; assume the result for $n = k$; add the $(k+1)$th term and show you get the formula with $k+1$ in place of n; write the conclusion.

YOUR TURN · WE DO

B1 Prove by induction that $\sum_{r=1}^{n}(2r-1) = n^2$ for all positive integers n. [5 marks]

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EXAMPLE 2 · I DO

Prove by induction that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Hint In the step, take out the common factor $(k+1)(k+2)$ before you expand anything.

YOUR TURN · WE DO

B2 Prove by induction that $\sum_{r=1}^{n} r(r+3) = \frac{n(n+1)(n+5)}{3}$ for all positive integers n. [5 marks]

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C1 [4 marks]

Alex proves that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$. The base case and the algebra in the step are correct. Alex ends: "So it is true for $n = 1$, $n = k$ and $n = k + 1$, so it is true for all n." Explain what is wrong with this conclusion, and write a correct one.

C2 [5 marks]

Prove by induction that $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n.

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PART 2 Sums of fractions

EXAMPLE 3 · I DO

Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all positive integers n.

Hint In the step, put the two fractions over the denominator $(k+1)(k+2)$. The numerator factorises.

YOUR TURN · WE DO

B3 Prove by induction that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for all positive integers n. [5 marks]

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C3 [5 marks]

Prove by induction that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$ for all positive integers n.

C4 [6 marks]

Let $S_n = \sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$. Find S_1, S_2 and S_3, write down a formula for S_n, and prove it by induction.

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PART 3 Sums of powers

EXAMPLE 4 · I DO

Prove by induction that $\sum_{r=0}^{n} 2^r = 2^{n+1} - 1$ for all non-negative integers n.

Hint The sum starts at $r = 0$, so the base case is $n = 0$. In the step, $2^{k+1} + 2^{k+1} = 2 \times 2^{k+1} = 2^{k+2}$.

YOUR TURN · WE DO

B4 Prove by induction that $\sum_{r=0}^{n} 5^r = \frac{5^{n+1} - 1}{4}$ for all non-negative integers n. [5 marks]

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EXAMPLE 5 · I DO

Prove by induction that $\sum_{r=1}^{n} r \times 2^r = (n-1)2^{n+1} + 2$ for all positive integers n.

Hint The $(k+1)$th term is $(k+1)2^{k+1}$. Collect the terms in 2^{k+1}, then use $2k \times 2^{k+1} = k \times 2^{k+2}$.

YOUR TURN · WE DO

B5 Prove by induction that $\sum_{r=1}^{n} r \times 3^{r-1} = \frac{(2n-1)3^n + 1}{4}$ for all positive integers n. [6 marks]

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C5 [5 marks]

The result $\sum_{r=1}^{n}(r+1)2^{r-1} = n \times 2^n$ is to be proved. Sam writes:

"Assume it is true for $n = k+1$: $\sum_{r=1}^{k+1}(r+1)2^{r-1} = (k+1)2^{k+1}$.

Take away the last term to get the sum to k." Explain what is wrong with Sam's method, and write a correct step from $n = k$ to $n = k+1$.

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C6 [7 marks]

Prove by induction that $\sum_{r=0}^{n}(2r+1)2^r = (2n-1)2^{n+1}+3$ for all non-negative integers n.

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D1

> Jo writes: "If it is true for $n = k$, the sum to $k + 1$ is $\frac{(2k+1)^2}{8} + (k + 1) = \frac{(2k+3)^2}{8}$. So it is true for $n = k + 1$, and so it is true for all n."

Jo claims that $\sum_{r=1}^{n} r = \frac{(2n+1)^2}{8}$ for all positive integers n.

(a) Show that the algebra in Jo's step is correct. [3 marks]

(b) Explain why Jo has not proved the claim. [2 marks]

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(c) Jo's step still works if the formula is $\frac{(2n+1)^2}{8} + c$ for a constant c. Find the value of c that makes the formula true for $n = 1$, and show that the formula is then $\frac{n(n+1)}{2}$. [3 marks]

Answer .....

Total for D1: 8 marks

D2 It is given that $\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{n(n+1)^2(n+2)}{2}$.

(a) Show that $\frac{k(k+1)^2(k+2)}{2} + (k+1)(k+2)(2k+3) = \frac{(k+1)(k+2)^2(k+3)}{2}$. [4 marks]

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(b) Hence prove the formula by induction. [5 marks]

Total for D2: 9 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 [8 marks]

Prove by induction that $\prod_{r=2}^{n}\left(1-\frac{1}{r^2}\right)=\frac{n+1}{2n}$ for all integers $n \geqslant 2$, where $\prod$ means the product of the terms. Hence find the exact value of $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{99^2}\right)$.

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 A product is built by multiplying on the next factor, so the step multiplies, it does not add.

Hint 2 Write $1 - \dfrac{1}{(k+1)^2}$ as one fraction and factorise its numerator, a difference of two squares.

Hint 3 The base case is $n = 2$, so the conclusion is for integers $n \geqslant 2$. For the last part, the product stops at $r = 99$.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can prove a formula for a sum by induction, writing the base case, the assumption, the step and the conclusion.	○ ○ ○
I can add the $(k+1)$th term and rearrange to the form the formula gives for $n = k + 1$.	○ ○ ○
I can prove a result for non-negative integers, starting from $n = 0$.	○ ○ ○
I can spot a mistake in an induction proof and write a correct conclusion.	○ ○ ○

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