Invariant points and invariant lines

89 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-invariance/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Work out $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ 2x+1 \end{pmatrix}$.

Answer .....

A2 Needed today [2 marks]

Show that the equations $2x - 3y = 0$ and $-4x + 6y = 0$ describe the same line, and write it in the form $y = mx$.

Answer .....

A3 Last lesson [2 marks]

Write down the matrix for a reflection in the line $y = -x$, and the image of $(2, 5)$.

Answer .....

A4 From A-Level Maths [1 mark]

Write down the gradient of a line perpendicular to $y = \frac{2}{3}x$.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 Invariant points

EXAMPLE 1 · I DO

Find the invariant points of the transformation with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

Hint A point is invariant when its image is itself: solve $M\mathbf{x} = \mathbf{x}$, and use BOTH equations.

YOUR TURN · WE DO

B1 Find the invariant points of the transformation with matrix $\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}$. [4 marks]

Answer .....

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EXAMPLE 2 · I DO

Show that the transformation with matrix $\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$ has a line of invariant points, and find its equation.

Hint If the two equations from $M\mathbf{x} = \mathbf{x}$ are the same equation, every point on that line is invariant.

YOUR TURN · WE DO

B2 Show that the transformation with matrix $\begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}$ has a line of invariant points, and find its equation. [4 marks]

Answer .....

PRACTICE · YOU DO

C1 Show that the transformation with matrix $\begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$ has a line of invariant points, and find its equation. [4 marks]

Answer .....

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C2 $M = \begin{pmatrix} k & 4 \\ 1 & -1 \end{pmatrix}$ has a line of invariant points. Find k and the equation of the line. [5 marks]

Answer .....

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PART 2 Invariant lines through the origin

EXAMPLE 3 · I DO

Find the invariant lines through the origin of the transformation with matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Hint A line $y = mx$ is invariant if the image of $(1, m)$ lies on the same line.

YOUR TURN · WE DO

B3 Find the invariant lines through the origin of the transformation with matrix $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$. [5 marks]

Answer .....

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C3 Show that a rotation of 90° anticlockwise about the origin has no invariant lines through the origin, and only one invariant point. [4 marks]

C4 $M = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$. Sam says: "$(3t, t) \to (9t, 3t)$, so $y = \frac{1}{3}x$ is a line of invariant points." Explain Sam's mistake, and find the line of invariant points of M. [4 marks]

Answer .....

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C5 Find the invariant lines through the origin of the transformation with matrix $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. [5 marks]

Answer .....

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PART 3 Invariant lines not through the origin

EXAMPLE 4 · I DO

Show that the line $y = x + 1$ is an invariant line of the transformation with matrix $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$.

Hint Take a general point $(t, t+1)$ of the line, find its image, and check that the image is on the same line.

YOUR TURN · WE DO

B4 Show that the line $y = 2x + 3$ is an invariant line of the transformation with matrix $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. [3 marks]

Answer .....

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EXAMPLE 5 · I DO

$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Show that every line $y = x + c$ is invariant. Is every line $y = -x + c$ invariant?

Hint Take a general point $(t, t + c)$ on the line, find its image (X, Y), and look for an equation linking X and Y.

YOUR TURN · WE DO

B5 $M = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$. Find the line of invariant points of M. Show that every line $y = \frac{1}{2}x + c$ is an invariant line. [6 marks]

Answer .....

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C6 Alex says that the stretch $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ has only two invariant lines, $y = 0$ and $x = 0$. Find all the invariant lines, and say which is a line of invariant points. [5 marks]

Answer .....

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C7 $M = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$. Find the line of invariant points of M, and show that every line $y = -x + c$ is an invariant line. [6 marks]

Answer .....

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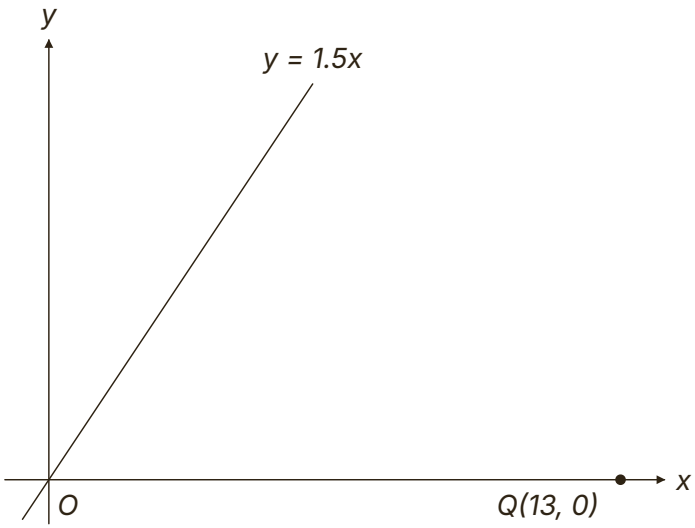
D1

Alex says: "The only invariant line of M is $y = \frac{3}{2}x$."

$M = \begin{pmatrix} -\frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$ represents a reflection in the line $y = \frac{3}{2}x$.

(a) Show that every point on $y = \frac{3}{2}x$ is invariant. [3 marks]



(b) Show that Alex is wrong by finding all the other invariant lines. [3 marks]

Answer .....

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(c) Find the image Q' of the point $Q(13, 0)$, and show that the midpoint of QQ' lies on the mirror and QQ' is perpendicular to it. [4 marks]

Total for D1: 10 marks

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D2 In this question you must show detailed reasoning. Find every invariant line of the transformation with matrix $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$, of the form $y = mx + c$, and say which of them is a line of invariant points. [8 marks]

Answer .....

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The transformation T has the line $y = 2x$ as a line of invariant points, and maps the point $(1, -1)$ to $(3, -3)$. Find the matrix of T. Find all the invariant lines of T, and say which are lines of invariant points. [9 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Two facts give you the images of two points: $(1, 2)$ stays where it is, and $(1, -1)$ goes to $(3, -3)$.

Hint 2 Put the two points side by side as the columns of one matrix, and their images as the columns of another. Then use an inverse.

Hint 3 For the invariant lines, find the image of $(t, mt + c)$ and ask when it lies on the same line for every t.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can find the invariant points of a transformation, including a line of invariant points.	○ ○ ○
I can find the invariant lines of a transformation through the origin.	○ ○ ○
I can show that a line not through the origin, or a whole family of lines, is invariant.	○ ○ ○
I can tell an invariant line from a line of invariant points.	○ ○ ○

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