Quartic equations and symmetric functions of roots

93 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-roots-quartic/

Date

A Do Now

Quick questions to start.

A1 Last lesson [3 marks]

The roots of $2x^3 - 5x^2 + 3x - 7 = 0$ are α, β and γ. Write down $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

Answer .....

A2 Needed today [2 marks]

Expand $(a + b + c)^2$.

Answer .....

A3 Last lesson [2 marks]

The roots of $x^2 - 3x + 5 = 0$ are α and β. Find $\alpha^2 + \beta^2$.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

PART 1 The sums for a quartic

EXAMPLE 1 · I DO

The roots of $2x^4 - 3x^3 + 5x^2 + x - 4 = 0$ are α, β, γ and δ. Write down $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ and $\alpha\beta\gamma\delta$. ($\sum \alpha\beta$ is the sum over all six pairs, $\sum \alpha\beta\gamma$ over all four triples.)

Hint For $ax^4 + bx^3 + cx^2 + dx + e = 0$ the signs go $-, +, -, +$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\sum \alpha\beta\gamma = -\frac{d}{a}$, $\alpha\beta\gamma\delta = \frac{e}{a}$.

YOUR TURN · WE DO

B1 The roots of $3x^4 + 6x^3 - x^2 + 4x + 9 = 0$ are α, β, γ and δ. Write down $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ and $\alpha\beta\gamma\delta$, each in its simplest form. [4 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

EXAMPLE 2 · I DO

The roots of $x^4 - 2x^3 + 4x^2 + x - 3 = 0$ are α, β, γ and δ. Find $\sum \alpha^2$, and explain what it tells you about the roots.

Hint $(\sum \alpha)^2$ gives every square once and every pair twice, so $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$.

YOUR TURN · WE DO

B2 The roots of $2x^4 + 6x^3 + 11x^2 - 3x + 5 = 0$ are α, β, γ and δ. Find $\sum \alpha^2$. Hence explain why the equation has at least two roots that are not real. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

C1 The roots of $x^4 + kx^3 + 6x^2 - 4x + 2 = 0$ are α, β, γ and δ, where k is real. Given that $\sum \alpha^2 = 13$, find the possible values of k. [4 marks]

Answer .....

C2 The roots of $3x^4 - 2x^3 + 7x^2 + 5x - 6 = 0$ are α, β, γ and δ. Sam writes: "$\sum \alpha = -\frac{2}{3}$ and $\alpha\beta\gamma\delta = 2$." Explain each of Sam's mistakes, and find $\sum \alpha^2$. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

PART 2 Reciprocals and products of pairs

EXAMPLE 3 · I DO

The roots of $x^4 + 3x^3 - 2x^2 + 5x - 6 = 0$ are α, β, γ and δ. Find $\sum \frac{1}{\alpha}$ and $\sum \frac{1}{\alpha^2}$.

Hint Put $\sum \frac{1}{\alpha}$ over the common denominator $\alpha\beta\gamma\delta$. Then use the squares rule on the numbers $\frac{1}{\alpha}$.

YOUR TURN · WE DO

B3 The roots of $2x^4 - x^3 + 4x^2 - 6x + 3 = 0$ are α, β, γ and δ. Find $\sum \frac{1}{\alpha}$ and $\sum \frac{1}{\alpha^2}$. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

EXAMPLE 4 · I DO

The roots of $x^3 - 4x^2 + 2x + 5 = 0$ are α, β and γ. Find $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$.

Hint Square $\alpha\beta + \beta\gamma + \gamma\alpha$. The cross terms are $2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)$, and that bracket has a common factor.

YOUR TURN · WE DO

B4 The roots of $2x^3 + x^2 - 6x + 4 = 0$ are α, β and γ. Find $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$. [5 marks]

Answer .....

PRACTICE · YOU DO

C3 In this question you must show detailed reasoning. The roots of $x^3 - 5x^2 + 3x - 2 = 0$ are α, β and γ. Find the value of $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$. [4 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

C4 The roots of $x^3 + 2x^2 - 3x + 5 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

PART 3 Relationships between the roots

EXAMPLE 5 · I DO

The roots of $x^3 - 3x^2 + x + 4 = 0$ are α, β and γ. Use the equation to find $\alpha^3 + \beta^3 + \gamma^3$.

Hint Each root satisfies the equation, so $\alpha^3 = 3\alpha^2 - \alpha - 4$. Write this for all three roots and add.

YOUR TURN · WE DO

B5 The roots of $2x^3 + x^2 - 4x + 3 = 0$ are α, β and γ. Use the equation to find $\alpha^3 + \beta^3 + \gamma^3$. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

EXAMPLE 6 · I DO

The roots of $x^4 - 8x^3 + px^2 + qx + r = 0$ form an arithmetic sequence with common difference 2. Find the roots, and the values of p, q and r.

Hint Call the roots $a - 3$, $a - 1$, $a + 1$, $a + 3$. Their sum gives a.

YOUR TURN · WE DO

B6 The roots of $x^4 - 12x^3 + px^2 + qx + r = 0$ form an arithmetic sequence with common difference 4. Find the roots, and the values of p, q and r. [6 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

C5 The roots of $2x^4 - 3x^2 + 4x + 6 = 0$ are α, β, γ and δ. Find $\sum \alpha^2$, and use the equation to find $\sum \alpha^4$. [5 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

C6 In this question you must show detailed reasoning. The equation $x^4 - 6x^3 + px^2 + qx - 72 = 0$ has roots α, $-\alpha$, β and 2β, where α and β are real and $\alpha > 0$. Find the four roots and the values of p and q. [7 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

D1 Ali writes: "$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha) = 36 - k$."

The roots of $x^3 - 6x^2 + kx - 10 = 0$ are α, β and γ, where k is a constant.

(a) Explain Ali's mistake. [2 marks]

(b) Given that $\alpha^2 + \beta^2 + \gamma^2 = 20$, find k. [2 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

(c) Use the equation to find $\alpha^3 + \beta^3 + \gamma^3$. [4 marks]

Answer .....

Total for D1: 8 marks

D2 The equation $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0$ has roots α, β, γ and δ, where $\alpha + \beta = \gamma + \delta$.

(a) Show that $\alpha + \beta = 3$. [2 marks]

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

(b) Let $m = \alpha\beta$ and $n = \gamma\delta$. Show that $m + n = -2$ and find mn. [4 marks]

Answer .....

(c) Hence solve the equation. [4 marks]

Answer .....

Total for D2: 10 marks

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The roots of $x^3 + px^2 + qx + r = 0$, where p and r are non-zero, form a geometric sequence. Show that $q^3 = p^3 r$. Use this to show that the roots of $x^3 - 7x^2 + 14x - 8 = 0$ form a geometric sequence, and find them. [8 marks]

Answer .....

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.

Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Call the roots $\frac{a}{t}$, a and at. Their product is short.

Hint 2 The sum of the roots and the sum of their pairs share a factor. Divide one by the other.

Hint 3 For the example, the middle root is $-\frac{q}{p}$. Then use the sum of the roots to find t.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can write down $\sum\alpha$, $\sum\alpha\beta$, $\sum\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$ for a quartic, with the right signs. ○ ○ ○

I can find sums of squares, reciprocals and cubes of the roots from the coefficients. ○ ○ ○

I can use the equation itself to find a sum of cubes or fourth powers. ○ ○ ○

I can use a link between the roots to find them, or to find an unknown coefficient. ○ ○ ○

© 2026 Mathswariz. All rights reserved. Free for personal and classroom use. Not to be sold or republished.