Equations whose roots are related by a substitution

91 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-roots-substitution/

Date

A Do Now

Quick questions to start.

A1 Needed today [2 marks]

Expand and simplify $(u-2)^3$.

Answer .....

A2 Needed today [2 marks]

Simplify $(2\sqrt{u})^2$ and $(\sqrt{u}-3)^2$.

Answer .....

A3 Last lesson [2 marks]

The roots of $x^3 - 4x^2 + x - 3 = 0$ are α, β and γ. Find $\alpha^2 + \beta^2 + \gamma^2$.

Answer .....

A4 From A-Level Maths [2 marks]

Make $\sqrt{u}$ the subject of $u\sqrt{u} + 3\sqrt{u} = 2u + 5$.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 Moving and stretching the roots

EXAMPLE 1 · I DO

The roots of $x^3 - 4x^2 + 2x - 5 = 0$ are α, β and γ. Use the substitution $u = x + 1$ to find a cubic equation whose roots are $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

Hint Write x in terms of u first: $x = u - 1$. Put that into the equation and expand.

YOUR TURN · WE DO

B1 The roots of $2x^3 + x^2 - 3x + 4 = 0$ are α, β and γ. Use a substitution to find a cubic equation, with integer coefficients, whose roots are $\alpha - 2$, $\beta - 2$ and $\gamma - 2$. [5 marks]

Answer .....

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EXAMPLE 2 · I DO

The roots of $x^3 - x^2 + 3x - 2 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$.

Hint $u = 2x - 1$, so $x = \dfrac{u+1}{2}$. The denominators are 8, 4 and 2: multiply through by 8.

YOUR TURN · WE DO

B2 The roots of $2x^3 + 3x^2 - x + 1 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $3\alpha + 1$, $3\beta + 1$ and $3\gamma + 1$. [5 marks]

Answer .....

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C1 The roots of $x^3 + 2x^2 - x + 4 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are 3α, 3β and 3γ. [3 marks]

Answer .....

C2 In this question you must show detailed reasoning. By using the substitution $u = x - 1$, solve the equation $x^3 - 3x^2 + 3x + 7 = 0$. [5 marks]

Answer .....

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PART 2 Reciprocals and squares of the roots

EXAMPLE 3 · I DO

The roots of $2x^3 - 3x^2 + x + 5 = 0$ are α, β and γ. Find a cubic equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Hint $u = \frac{1}{x}$, so $x = \frac{1}{u}$. After substituting, multiply through by u^3 to clear the fractions.

YOUR TURN · WE DO

B3 The roots of $x^4 + 2x^3 - 4x + 3 = 0$ are α, β, γ and δ. Find a quartic equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$. [5 marks]

Answer .....

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EXAMPLE 4 · I DO

The roots of $x^3 + 2x^2 - 3x + 1 = 0$ are α, β and γ. Find a cubic equation whose roots are α^2, β^2 and γ^2.

Hint Put $x = \sqrt{u}$. Then $x^3 = u\sqrt{u}$ and $x^2 = u$. Collect the terms with $\sqrt{u}$ on one side, then square both sides.

YOUR TURN · WE DO

B4 The roots of $2x^3 - x^2 + 4x - 3 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are α^2, β^2 and γ^2. [5 marks]

Answer .....

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C3 The roots of $x^3 + 4x^2 - 2x + 3 = 0$ are α, β and γ. Use a substitution to find a cubic equation, with integer coefficients, whose roots are $\frac{1}{\alpha + 1}$, $\frac{1}{\beta + 1}$ and $\frac{1}{\gamma + 1}$. [5 marks]

Answer .....

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C4 The roots of $x^3 - x^2 + 4x - 2 = 0$ are α, β and γ. Sam wants an equation with roots α^2, β^2 and γ^2. Sam writes: "Put $x = \sqrt{u}$: $u\sqrt{u} - u + 4\sqrt{u} - 2 = 0$. Square each term: $u^3 - u^2 + 4u - 4 = 0$." Explain what Sam has done wrong, and find the correct equation. [5 marks]

Answer .....

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PART 3 Roots written in terms of one root

EXAMPLE 5 · I DO

The roots of $x^3 - 2x^2 + 3x + 4 = 0$ are α, β and γ. Find a cubic equation whose roots are $\beta + \gamma$, $\gamma + \alpha$ and $\alpha + \beta$.

Hint $\alpha + \beta + \gamma = 2$, so $\beta + \gamma = 2 - \alpha$. Each new root is $2 - x$ for an old root x: put $u = 2 - x$, so $x = 2 - u$.

YOUR TURN · WE DO

B5 The roots of $x^3 + x^2 - 2x + 5 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $\beta + \gamma$, $\gamma + \alpha$ and $\alpha + \beta$. [5 marks]

Answer .....

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EXAMPLE 6 · I DO

The roots of $x^3 - 2x^2 + 4x - 3 = 0$ are α, β and γ. Find a cubic equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

Hint $\alpha\beta\gamma = 3$, so $\alpha\beta = \dfrac{3}{\gamma}$. Each new root is $\dfrac{3}{x}$ for an old root x: put $x = \dfrac{3}{u}$.

YOUR TURN · WE DO

B6 The roots of $x^3 + 3x^2 - x + 2 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$. [6 marks]

Answer .....

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C5 The roots of $x^3 - 3x^2 + 2x - 1 = 0$ are α, β and γ. Use $\alpha\beta\gamma$ to find a cubic equation, with integer coefficients, whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. [4 marks]

Answer .....

C6 The roots of $x^3 - 5x^2 + 2x + 4 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $\alpha + \beta$, $\beta + \gamma$ and $\gamma + \alpha$, and hence find the value of $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$. [5 marks]

Answer .....

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D Exam-style questions

OCR exam style

D1 In this question you must show detailed reasoning. The equation $x^4 + 4x^3 - x^2 - 10x + 4 = 0$ is to be solved.

(a) Show that the substitution $x = w - 1$ turns the equation into $w^4 - 7w^2 + 10 = 0$. [4 marks]

(b) Hence solve $x^4 + 4x^3 - x^2 - 10x + 4 = 0$, giving each root in exact form. [5 marks]

Answer .....

Total for D1: 9 marks

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D2

> Ben writes: “The new roots are twice the old ones, so I put $x = 2u$.”

The roots of $2x^3 - 3x^2 + 5x - 1 = 0$ are α, β and γ. An equation whose roots are 2α, 2β and 2γ is wanted.

(a) Explain why Ben's substitution is wrong. [2 marks]

(b) Use a substitution to find a cubic equation, with integer coefficients, whose roots are 2α, 2β and 2γ. [3 marks]

Answer .....

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(c) Hence find the value of $(2\alpha)^2 + (2\beta)^2 + (2\gamma)^2$. [3 marks]

Answer .....

Total for D2: 8 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 The roots of $x^3 - 3x^2 + 5 = 0$ are α, β and γ. Find a cubic equation, with integer coefficients, whose roots are $\dfrac{\alpha}{\beta+\gamma}$, $\dfrac{\beta}{\gamma+\alpha}$ and $\dfrac{\gamma}{\alpha+\beta}$. Hence find $\dfrac{\alpha}{\beta+\gamma} + \dfrac{\beta}{\gamma+\alpha} + \dfrac{\gamma}{\alpha+\beta}$. [8 marks]

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Use $\alpha + \beta + \gamma = 3$ to write $\beta + \gamma$ in terms of α alone.

Hint 2 Each new root is $u = \dfrac{x}{3 - x}$. Rearrange to make x the subject.

Hint 3 With $x = \dfrac{3u}{1 + u}$, multiply the whole equation by $(1 + u)^3$ before expanding.

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can find an equation whose roots are $\alpha + k$, $k\alpha$ or $\dfrac{1}{\alpha}$ by substituting for x.	○ ○ ○
I can find an equation whose roots are the squares of the roots, by putting $\sqrt{u}$ on one side and squaring.	○ ○ ○
I can write roots such as $\beta + \gamma$ or $\alpha\beta$ in terms of one root, and choose the substitution.	○ ○ ○
I can use the new equation to solve the old one, or to find a sum of powers of the roots.	○ ○ ○

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