Transformations in three dimensions

98 marks

Hints online, answers free with an account: mathswariz.uk/worksheets/ocr-further-maths-transform-3d/

Date

A Do Now

Quick questions to start.

A1 Last lesson [2 marks]

$\det A = 4$ and $\det B = -3$. A shape of area 2 is transformed by A and then by B. Find the area of the final image.

Answer .....

A2 From chapter 3 [3 marks]

Find the determinant of $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix}$.

Answer .....

A3 Needed today [2 marks]

Work out $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

Answer .....

A4 From A-Level Maths [2 marks]

Write down the exact values of $\cos 150°$ and $\sin 150°$.

Answer .....

B Examples and practice

A-Level Further

Your teacher works each example with you (I do). You try the one beside it together (We do). Then practise that part on your own (You do).

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PART 1 Matrices for reflections and rotations

EXAMPLE 1 · I DO

Find the matrix for a reflection in the plane $y = 0$, and the image of the point $(2, 3, -1)$.

Hint A reflection in $y = 0$ changes the sign of the y-coordinate and leaves x and z alone. The columns are the images of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

YOUR TURN · WE DO

B1 Find the matrix for a reflection in the plane $z = 0$. Find the image of $(4, -2, 5)$, and the point whose image is $(1, 7, -3)$. [4 marks]

Answer .....

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EXAMPLE 2 · I DO

Find the matrix for a rotation of $90°$ anticlockwise about the z-axis, and the image of $(1, 2, 3)$.

Hint The rotation about the z-axis is $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$: the xy-plane turns as in two dimensions and z stays the same.

YOUR TURN · WE DO

B2 Find the matrix for a rotation of $90°$ anticlockwise about the x-axis, and the image of $(3, 1, 4)$. [4 marks]

Answer .....

PRACTICE · YOU DO

C1 R is the rotation of $60°$ anticlockwise about the z-axis. The point P has image $(\sqrt{3}, 1, 4)$ under R. Find P. [4 marks]

Answer .....

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C2 A rotation about one of the coordinate axes maps the point $(3, -1, 2)$ to $(3, 2, 1)$. Find the axis, the angle and direction of the rotation, and its matrix. Find the image of $(0, 4, -3)$. [8 marks]

Answer .....

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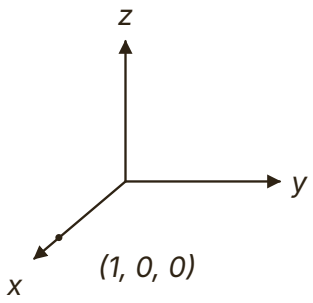
PART 2 The y-axis rotation and describing a transformation

EXAMPLE 3 · I DO

Find the matrix for a rotation of $90°$ about the y-axis, and the image of $(1, 0, 0)$.



Hint About the y-axis the matrix is $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$. The signs of $\sin\theta$ are the other way round from the x and z rotations.

YOUR TURN · WE DO

B3 Find the matrix for a rotation of $270°$ anticlockwise about the y-axis, and the image of $(2, -1, 3)$. [5 marks]

Answer .....

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EXAMPLE 4 · I DO

Describe fully the transformations represented by (i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

Hint Find which coordinate axis or plane is left alone. For a rotation, read $\cos\theta$ and $\sin\theta$ from the matrix and give the angle and its direction.

YOUR TURN · WE DO

B4 Describe fully the transformations represented by (i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$. [5 marks]

Answer .....

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C3 $N = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$. Show that $N = 2R$, where R is the matrix of a rotation about a coordinate axis. Describe fully the transformation N represents. [3 marks]

Answer .....

C4 R is a rotation of 90° anticlockwise about the z-axis and X is a reflection in the plane $y = 0$.

(a) Write down the matrices R and X. [2 marks]

Answer .....

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(b) Find the matrix for X followed by R, and the matrix for R followed by X. [4 marks]

Answer .....

(c) Describe fully the single transformation X followed by R, giving a reason. [3 marks]

Answer .....

Total for C4: 9 marks

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PART 3 Volume, orientation and singular matrices

EXAMPLE 5 · I DO

A cube of side 2 is transformed by $M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Find the volume of the image.

Hint In three dimensions the determinant is the volume scale factor. Its sign tells you whether orientation is kept.

YOUR TURN · WE DO

B5 [5 marks]

$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & k & 1 \\ 3 & 0 & 1 \end{pmatrix}$. A shape of volume 5 is mapped by M to a shape of volume 15, and the orientation is reversed. Find k.

Answer .....

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C5 A solid of volume 4 is transformed by a matrix with determinant -5. Sam says the image has volume -20. Alex says it has volume $5^3 \times 4 = 500$. Explain what each has done wrong, and give the volume of the image and the effect on orientation. [4 marks]

Answer .....

C6 $M = \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 1 \\ x & 0 & 2 \end{pmatrix}$. Find the exact values of x for which the images under M of O, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ all lie in one plane. [5 marks]

Answer .....

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C7 $M = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & a \end{pmatrix}$, where a is a real constant.

(a) Show that $\det M = a^3 + 1$. [2 marks]

(b) A solid of volume 3 is mapped by M to a solid of volume 27, and the orientation is kept. Find a. [3 marks]

Answer .....

(c) Find the value of a for which M is singular. [2 marks]

Answer .....

Total for C7: 7 marks

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D1

> Alex says: "The matrix of T is $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$."

The transformation T is a rotation of 90° anticlockwise about the y-axis.

(a) By finding the image of $(1, 0, 0)$ under Alex's matrix and thinking about the direction of the turn, show that Alex is wrong. [3 marks]

(b) Write down the correct matrix for T, and find the image of $(2, 5, -1)$. [3 marks]

Answer .....

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(c) Describe the transformation that Alex's matrix does represent. [2 marks]

Answer .....

Total for D1: 8 marks

D2

In this question you must show detailed reasoning. $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Show that $\det M = 6x - 5$. [2 marks]

(b) Given $x = 4$, find the volume of the image of a solid of volume 6, and say whether the orientation is kept. [3 marks]

Answer .....

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(c) Find the value of x for which the image of every point lies in one plane. [3 marks]

Answer .....

(d) Given $x = 0$, a cuboid of volume 6 is transformed by M. Find the volume of the image and the effect on orientation. [2 marks]

Answer .....

Total for D2: 10 marks

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E Extension

Stretch

No route is given. The hints are at the end of the sheet. Use one at a time.

E1 [8 marks]

$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Show that P is not a reflection in a coordinate plane or a rotation about a coordinate axis. Show that P is a rotation of 90° about the x-axis followed by a rotation of 90° about the z-axis, both anticlockwise. Find the smallest positive integer n with $P^n = I$, and explain it in terms of what P does to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Answer .....

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Extension hints

Cover these with paper. Uncover one at a time.

E1

Hint 1 Read the columns: the first column is where $\mathbf{i}$ goes. Does any one of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ stay where it is?

Hint 2 Write down the two rotation matrices. The rotation done first goes on the right of the product.

Hint 3 Follow $\mathbf{i}$ through P, then P again, then P again. When is it back where it started?

How did it go?

Colour one circle on each line: not yet, nearly, yes.

I can write down the matrix for a reflection in a coordinate plane or a rotation about a coordinate axis.	○ ○ ○
I can describe fully the transformation a 3 by 3 matrix represents.	○ ○ ○
I can use the determinant as the volume scale factor and say whether orientation is kept.	○ ○ ○
I can use a zero determinant to show that the images of points lie in a plane.	○ ○ ○

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